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INTRODUCTION TO SUPERSYMMETRY

JOSEPH D. LYKKEN
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, IL 60510

These lectures give a self-contained introduction to supersymmetry from a modern perspective. Emphasis is placed on material essential to understanding duality. Topics include: central charges and BPS-saturated states, supersymmetric nonlinear sigma models, N=2 Yang-Mills theory, holomorphy and the N=2 Yang-Mills β function, supersymmetry in 2, 6, 10, and 11 spacetime dimensions.

1 Introduction

> *"Never mind, lads. Same time tomorrow. We must get a winner one day."*
> – Peter Cook, as the doomsday prophet in "The End of the World".

Supersymmetry, along with its monozygotic sibling superstring theory, has become the dominant framework for formulating physics beyond the standard model. This despite the fact that, as of this morning, there is no unambiguous experimental evidence for either idea. Theorists find supersymmetry appealing for reasons which are both phenomenological and technical. In these lectures I will focus exclusively on the technical appeal. There are many good recent reviews of the phenomenology of supersymmetry. [1] Some good technical reviews are Wess and Bagger, [2] West, [3] and Sohnius. [4]

The goal of these lectures is to provide the student with the technical background requisite for the recent applications of duality ideas to supersymmetric gauge theories and superstrings. More specifically, if you absorb the material in these lectures, you will understand Section 2 of Seiberg and Witten, [5] and you will have a vague notion of why there might be such a thing as M-theory. Beyond that, you're on your own.

2 Representations of Supersymmetry

2.1 The general 4-dimensional supersymmetry algebra

A symmetry of the S-matrix means that the symmetry transformations have the effect of merely reshuffling the asymptotic single and multiparticle states. The known symmetries of the S-matrix in particle physics are:

- Poincaré invariance, the semi-direct product of translations and Lorentz rotations, with generators P_m, M_{mn}.
- So-called "internal" global symmetries, related to conserved quantum numbers such as electric charge and isospin. The symmetry generators are Lorentz scalars and generate a Lie algebra,

$$[B_\ell, B_k] = iC^j_{\ell k} B_j \quad , \tag{1}$$

  where the $C^j_{\ell k}$ are structure constants.
- Discrete symmetries: C, P, and T.

In 1967, Coleman and Mandula [6] provided a rigorous argument which proves that, given certain assumptions, the above are the only possible symmetries of the S-matrix. The reader is encouraged to study this classic paper and think about the physical and technical assumptions which are made there.

The Coleman-Mandula theorem can be evaded by weakening one or more of its assumptions. In particular, the theorem assumes that the symmetry algebra of the S-matrix involves only commutators. Weakening this assumption to allow anticommuting generators as well as commuting generators leads to the possibility of **supersymmetry**. Supersymmetry (or SUSY for short) is defined as the introduction of anticommuting symmetry generators which transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ (i.e. spinor) representations of the Lorentz group. Since these new symmetry generators are spinors, not scalars, supersymmetry is not an internal symmetry. It is rather an extension of the Poincaré spacetime symmetries. Supersymmetry, defined as the extension of the Poincaré symmetry algebra by anticommuting spinor generators, has an obvious extension to spacetime dimensions other than four: the Coleman-Mandula theorem, on the other hand, has no obvious extension beyond four dimensions.

In 1975, Haag, Lopuszański, and Sohnius [7] proved that supersymmetry is the only additional symmetry of the S-matrix allowed by this weaker set of assumptions. Of course, one could imagine that a further weakening of assumptions might lead to more new symmetries, but to date no physically compelling examples have been exhibited. [8] This is the basis of the strong but not unreasonable assertion that:

Supersymmetry is the only possible extension of the known spacetime symmetries of particle physics.

In four-dimensional Weyl spinor notation (see the Appendix) the N supersymmetry generators are denoted by Q^A_α, $A=1, \ldots N$. The most general four-dimensional supersymmetry algebra is given in the Appendix: here we will be content with checking some of the features of this algebra.

The anticommutator of the Q^A_α with their adjoints is:

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A_B \quad . \tag{2}$$

To see this, note the right-hand side of Eq. 2 must transform as $(\frac{1}{2}, \frac{1}{2})$ under the Lorentz group. The most general such object that can be constructed out of P_m, M_{mn}, and B_ℓ has the form:

$$\sigma^m_{\alpha\dot\beta} P_m C^A_B \quad ,$$

where the C^A_B are complex Lorentz scalar coefficients. Taking the adjoint of the left-hand side of Eq. 2, using

$$\begin{aligned} \left(\sigma^m_{\alpha\dot\beta}\right)^\dagger &= \sigma^m_{\beta\dot\alpha} \quad , \\ \left(Q^A_\alpha\right)^\dagger &= \overline{Q}^A_{\dot\alpha} \quad , \end{aligned} \tag{3}$$

tells us that C^A_B is a hermitian matrix. Furthermore, since $\{Q, \overline{Q}\}$ is a positive definite operator, C^A_B is a positive definite hermitian matrix. This means that we can always **choose** a basis for the Q^A_α such that C^A_B is proportional to δ^A_B. The factor of two in Eq. 2 is simply a convention.

The SUSY generators Q^A_α commute with the translation generators:

$$[Q^A_\alpha, P_m] = [\overline{Q}^A_{\dot\alpha}, P_m] = 0 \quad . \tag{4}$$

This is not obvious since the most general form consistent with Lorentz invariance is:

$$\begin{aligned} [Q^A_\alpha, P_m] &= Z^A_B \sigma_{\alpha\dot\beta m} \overline{Q}^{\dot\beta B} \\ [\overline{Q}^{\dot\alpha A}, P_m] &= \left(Z^A_B\right)^* Q^B_\beta \bar\sigma^{\dot\alpha\beta}_m \quad , \end{aligned} \tag{5}$$

where the Z^A_B are complex Lorentz scalar coefficients. Note we have invoked here the Haag, Lopuszański, Sohnius theorem which tells us that there are no $(\frac{1}{2}, 1)$ or $(1, \frac{1}{2})$ symmetry generators.

To see that the Z^A_B all vanish, the first step is to plug Eq. 5 into the Jacobi identity:

$$\left[[Q^A_\alpha, P_m], P_n\right] + (\text{cyclic}) = 0 \quad . \tag{6}$$

Using Eq. 240 this yields:

$$-4i\,(ZZ^*)^A_B\, \sigma^{\ \beta}_{mn\alpha} Q^B_\beta = 0 \quad , \tag{7}$$

which implies that the matrix ZZ^* vanishes.

This is not enough to conclude that Z^A_B itself vanishes, but we can get more information by considering the most general form of the anticommutator of two Q's:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} X^{AB} + \epsilon_{\beta\gamma}\sigma^{mn\gamma}_\alpha M_{mn} Y^{AB} \quad . \tag{8}$$

Here we have used the fact that the rhs must transform as $(0,0) + (1,0)$ under the Lorentz group. The spinor structure of the two terms on the rhs is antisymmetric/symmetric respectively under $\alpha \leftrightarrow \beta$, so the complex Lorentz scalar matrices X^{AB} and Y^{AB} are also antisymmetric/symmetric respectively.

Now we consider $\epsilon^{\alpha\beta}$ contracted on the Jacobi identity

$$[\{Q^A_\alpha, Q^B_\beta\}, P_m] + \{[P_m, Q^A_\alpha], Q^B_\beta\} - \{[Q^B_\beta, P_m], Q^A_\alpha\} = 0 \quad . \tag{9}$$

Since X^{AB} commutes with P_m, and plugging in Eqs. 2.5.232 and 233, the above reduces to

$$-4\left(Z^{AB} - Z^{BA}\right) P_m = 0 \quad , \tag{10}$$

and thus Z^A_B is symmetric. Combined with ZZ^*=0 this means that $ZZ^\dagger$=0, which implies that Z^A_B vanishes, giving Eq. 4.

Having established Eq. 4, the symmetric part of the Jacobi identity Eq. 9 implies that $M_{mn}Y^{AB}$ commutes with P_m, which can only be true if Y^{AB} vanishes. Thus:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} X^{AB} \quad . \tag{11}$$

The complex Lorentz scalars X^{AB} are called **central charges**: further manipulations with the Jacobi identities show that the X^{AB} commute with the Q^A_α, $\overline{Q}_{\dot\alpha A}$, and in fact generate an Abelian invariant subalgebra of the compact Lie algebra generated by B_ℓ. Thus we can write:

$$X^{AB} = a^{\ell AB}\, B_\ell \quad , \tag{12}$$

where the complex coefficients $a^{\ell AB}$ obey the intertwining relation Eq. 264.

2.2 The 4-dimensional N=1 supersymmetry algebra

The Appendix also contains the special case of the four-dimensional N=1 supersymmetry algebra. For N=1 the central charges X^{AB} vanish by antisymmetry, and the coefficients S_ℓ are real. The Jacobi identity for $[[Q,B],B]$ implies that the structure constants $C^k_{m\ell}$ vanish, so the internal symmetry algebra is Abelian. Starting with

$$\begin{aligned} [Q_\alpha, B_\ell] &= S_\ell Q_\alpha \\ [\overline{Q}_{\dot\alpha}, B_\ell] &= -S_\ell \overline{Q}_{\dot\alpha} \quad , \end{aligned} \tag{13}$$

it is clear that we can rescale the Abelian generators B_ℓ and write:

$$\begin{aligned} [Q_\alpha, B_\ell] &= Q_\alpha \\ [\overline{Q}_{\dot\alpha}, B_\ell] &= -\overline{Q}_{\dot\alpha} \quad . \end{aligned} \tag{14}$$

Clearly only one independent combination of the Abelian generators actually has a nonzero commutator with Q_α and $\overline{Q}_{\dot\alpha}$; let us denote this $U(1)$ generator by R:

$$\begin{aligned} [Q_\alpha, R] &= Q_\alpha \\ [\overline{Q}_{\dot\alpha}, R] &= -\overline{Q}_{\dot\alpha} \quad . \end{aligned} \tag{15}$$

Thus the N=1 SUSY algebra in general possesses an internal (global) $U(1)$ symmetry known as **R symmetry**. Note that the SUSY generators have R-charge +1 and -1, respectively.

2.3 SUSY Casimirs

Since we wish to characterize the irreducible representations of supersymmetry on asymptotic single particle states, we need to exhibit the Casimir operators. It suffices to do this for the N=1 SUSY algebra, as the extension to N>1 is straightforward.

Recall that the Poincaré algebra has two Casimirs: the mass operator $P^2 = P_m P^m$, with eigenvalues m^2, and the square of the Pauli-Ljubanski vector

$$W_m = \frac{1}{2}\epsilon_{mnpq} P^n M^{pq} \quad . \tag{16}$$

W^2 has eigenvalues $-m^2 s(s+1)$, s=0, $\frac{1}{2}$, 1, ... for massive states, and $W_m = \lambda P_m$ for massless states, where λ is the helicity.

For N=1 SUSY, P^2 is still a Casimir (since P commutes with Q and $\overline{Q}$), but W^2 is not (M does not commute with Q and $\overline{Q}$). The actual Casimirs are P^2 and C^2, where

$$\begin{aligned} C^2 &= C_{mn}C^{mn} \quad , \\ C_{mn} &= B_m P_n - B_n P_m \quad , \\ B_m &= W_m - \frac{1}{4}\overline{Q}_{\dot\alpha}\bar\sigma_m^{\dot\alpha\beta} Q_\beta \quad . \end{aligned} \tag{17}$$

This is easily verified using the commutators:

$$\begin{aligned} [W_m, Q_\alpha] &= -i\sigma_{mn\alpha}^{\ \ \ \beta} Q_\beta P^n \quad , \\ [\overline{Q}_{\dot\beta}\bar\sigma_m^{\dot\beta\gamma} Q_\gamma, Q_\alpha] &= -2P_m Q_\alpha + 4i\sigma_{nm\alpha}^{\ \ \ \beta} P^n Q_\beta \quad , \end{aligned} \tag{18}$$

which imply:

$$
\begin{aligned}
[C_{mn}, Q_\alpha] &= [B_m, Q_\alpha]P_n - [B_n, Q_\alpha]P_m \\
&= 0 \quad .
\end{aligned}
\tag{19}
$$

2.4 Classification of SUSY irreps on single particle states

We now have enough machinery to construct all possible irreducible representations of supersymmetry on asymptotic (on-shell) physical states. We begin with N=1 SUSY, treating the massive and massless states separately. Unlike the case of Poincaré symmetry, we do not have to consider tachyons – they are forbidden by the fact that $\{Q, \overline{Q}\}$ is positive definite.

N=1 SUSY, massive states

We analyze massive states from the rest frame $P_m = (m, \vec{0})$. We can write:

$$
\begin{aligned}
C^2 &= 2m^4 J_i J^i \quad , \\
J_i &\equiv S_i - \frac{1}{4m}\overline{Q}\bar{\sigma}_i Q \quad ,
\end{aligned}
\tag{20}
$$

where S_i is the spin operator and i is a spatial index: $i = 1, 2, 3$. Both S_i and $\bar{\sigma}_i^{\dot{\alpha}\beta}$ obey the $SU(2)$ algebra, so

$$
[J_i, J_j] = i\epsilon_{ijk} J_k \quad , \tag{21}
$$

and J^2 has eigenvalues $j(j+1)$, j equal integers or half-integers.

The commutator of J_i with either Q or $\overline{Q}$ is proportional to $\vec{P}$ and thus vanishes since we are in the rest frame. Q_α, $\overline{Q}_{\dot{\alpha}}$ are in fact two pairs of creation/annihilation operators which fill out the N=1 massive SUSY irrep of fixed m and j:

$$
\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2m\sigma^0_{\alpha\dot{\beta}} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{22}
$$

Given any state of definite $|m, j\rangle$ we can define a new state

$$
\begin{aligned}
|\Omega\rangle &= Q_1 Q_2 |m, j\rangle \quad , \\
Q_1 |\Omega\rangle &= Q_2 |\Omega\rangle \quad = \quad 0 \quad .
\end{aligned}
\tag{23}
$$

Thus $|\Omega\rangle$ is a Clifford vacuum state with respect to the fermionic annihilation operators Q_1, Q_2. Note that $|\Omega\rangle$ has degeneracy $2j+1$ since j_3 takes values $-j, \dots j$.

Acting on $|\Omega\rangle$, J_i reduces to just the spin operator S_i, so $|\Omega\rangle$ is actually an eigenstate of spin:

$$|\Omega\rangle = |m, s, s_3\rangle \quad . \tag{24}$$

Thus we can characterize all the states in the SUSY irrep by mass and spin.

It is convenient to define conventionally normalized creation/annihilation operators:

$$\begin{aligned} a_{1,2} &= \frac{1}{\sqrt{2m}} Q_{1,2} \quad , \\ a_{\dot{1},\dot{2}} &= \frac{1}{\sqrt{2m}} \overline{Q}_{\dot{1},\dot{2}} \quad . \end{aligned} \tag{25}$$

Then for a given $|\Omega\rangle$ the full massive SUSY irrep is:

$$\begin{aligned} &|\Omega\rangle \\ &a_1^\dagger\,|\Omega\rangle \\ &a_2^\dagger\,|\Omega\rangle \\ \frac{1}{\sqrt{2}} a_1^\dagger a_2^\dagger\,|\Omega\rangle &= -\frac{1}{\sqrt{2}} a_2^\dagger a_1^\dagger\,|\Omega\rangle \end{aligned} \tag{26}$$

There are a total of $4(2j+1)$ states in the massive irrep.

We compute the spin of these states by using the commutators:

$$[S_3, \begin{pmatrix} a_2^\dagger \\ a_1^\dagger \end{pmatrix}] = \frac{1}{2} \begin{pmatrix} a_2^\dagger \\ -a_1^\dagger \end{pmatrix} \quad . \tag{27}$$

Thus for $|\Omega\rangle = |m, j, j_3\rangle$ we get states of spin $s_3 = j_3,\ j_3-\frac{1}{2},\ j_3+\frac{1}{2},\ j_3$.

As an example, consider the $j=0$ or **fundamental** $N=1$ massive irrep. Since $|\Omega\rangle$ has spin zero there are a total of four states in the irrep, with spins $s_3 = 0,\ -\frac{1}{2},\ \frac{1}{2}$, and 0, respectively. Since the parity operation interchanges $a_1^\dagger$ with $a_2^\dagger$, one of the spin zero states is a pseudoscalar. Thus these four states correspond to one massive Weyl fermion, one real scalar, and one real pseudoscalar.

N=1 SUSY, massless states

We analyze massless states from the light-like reference frame $P_m = (E, 0, 0, E)$. In this case

$$C^2 = -2E^2(B_0 - B_3)^2 = -\frac{1}{2}E^2\overline{Q}_{\dot{2}}Q_2\overline{Q}_{\dot{2}}Q_2 = 0 \quad . \tag{28}$$

Also we have:

$$\begin{aligned}\{Q_{\dot 1}, \overline{Q}_{\dot 1}\} &= 4E \quad , \\ \{Q_{\dot 2}, \overline{Q}_{\dot 2}\} &= 0 \quad .\end{aligned} \tag{29}$$

We can define a vacuum state $|\Omega\rangle$ as in the massive case. However we notice from Eq. 29 that the creation operator $\overline{Q}_{\dot 2}$ makes states of zero norm:

$$\langle\Omega| Q_2 \overline{Q}_{\dot 2} |\Omega\rangle = 0 \quad . \tag{30}$$

This means that we can set $\overline{Q}_{\dot 2}$ equal to zero in the operator sense. Effectively there is just one pair of creation/annihilation operators:

$$a = \frac{1}{2\sqrt{E}} Q_1 \quad , \quad a^\dagger = \frac{1}{2\sqrt{E}} \overline{Q}_{\dot 1} \quad . \tag{31}$$

$|\Omega\rangle$ is nondegenerate and has definite helicity λ. The creation operator $a^\dagger$ transforms like $(0, \frac{1}{2})$ under the Lorentz group, thus it increases helicity by 1/2. The massless N=1 SUSY irreps each contain two states:

$$\begin{aligned} |\Omega\rangle \qquad & \text{helicity } \lambda \quad , \\ a^\dagger |\Omega\rangle \qquad & \text{helicity } \lambda + \frac{1}{2} \quad . \end{aligned} \tag{32}$$

However this is not a CPT eigenstate in general, requiring that we pair two massless SUSY irreps to obtain four states with helicities λ, $\lambda+\frac{1}{2}$, $-\lambda-\frac{1}{2}$, and $-\lambda$.

N>1 SUSY, no central charges, massless states

Here we have N creation operators $a^\dagger_A$. These generate a total of 2^N states in the SUSY irrep. The states have the form:

$$\frac{1}{\sqrt{n!}} a^\dagger_{A_1} \ldots a^\dagger_{A_n} |\Omega\rangle \quad , \tag{33}$$

with degeneracy given by the binomial coefficient $\binom{N}{n}$. Denoting the helicity of $|\Omega\rangle$ by λ, the helicities in the irrep are λ, $\lambda+\frac{1}{2}$, $\ldots \lambda+\frac{N}{2}$. This is not a CPT eigenstate except in the special case $\lambda = -N/4$. Examples of some of the more important irreps are given in Table 1.

Table 1: Examples of N>1 massless SUSY irreps (no central charge)

N=2											
Ω_0	helicity	0	$\frac{1}{2}$	1							Ω_0 and Ω_{-1} together make one N=2 on-shell vector multiplet.
	no. of states	1	2	1							
Ω_{-1}	helicity	-1	$-\frac{1}{2}$	0							
	no. of states	1	2	1							
$\Omega_{-\frac{1}{2}}$	helicity	$-\frac{1}{2}$	0	$\frac{1}{2}$							A massless N=2 on-shell hypermultiplet.
	no. of states	1	2	1							
N=4											
Ω_{-1}	helicity	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1					A massless N=4 on-shell vector multiplet.
	no. of states	1	4	6	4	1					
N=8											
Ω_{-2}	helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	An N=8 gravity multiplet.
	no. of states	1	8	28	56	70	56	28	8	1	

N>1 SUSY, no central charges, massive states

In this case we have 2N creation operators $(a^A_\alpha)^\dagger$. There are $2^{2N}(2j+1)$ states in a massive irrep. Consider, for example, the fundamental N=2 massive irrep:

Ω_0 :	spin	0	$\frac{1}{2}$	1
	no. of spin irreps	5	4	1
	total no. of states	5	8	3

There are a grand total of 16 states. Let us describe them in more detail:

1 state :	$\|\Omega\rangle$	1 spin 0 state
4 states :	$(a^{A}_{\alpha})^{\dagger}\|\Omega\rangle$	4 spin $\frac{1}{2}$ states
6 states :	$(a^{A_1}_{\alpha_1})^{\dagger}(a^{A_2}_{\alpha_2})^{\dagger}\|\Omega\rangle$	3 spin 1 and 3 spin 0 states
4 states :	$(a^{A_1}_{\alpha_1})^{\dagger}(a^{A_2}_{\alpha_2})^{\dagger}(a^{A_3}_{\alpha_3})^{\dagger}\|\Omega\rangle$	4 spin $\frac{1}{2}$ states
1 state :	$(a^{A_1}_{\alpha_1})^{\dagger}(a^{A_2}_{\alpha_2})^{\dagger}(a^{A_3}_{\alpha_3})^{\dagger}(a^{A_4}_{\alpha_4})^{\dagger}\|\Omega\rangle$	1 spin 0 state

The only counting which is not obvious is $6 = 3$ spin $1 + 3$ spin 0: this can be verified by looking at the Lorentz group tensor products:

$$\left[(0,\frac{1}{2})^1 \oplus (0,\frac{1}{2})^2\right] \otimes \left[(0,\frac{1}{2})^1 \oplus (0,\frac{1}{2})^2\right] =$$
$$(0,0) + \left[(0,1) + (0,0)\right] + (0,0) \quad .$$

The key point is that $(a^{A}_{\alpha})^{\dagger}(a^{A}_{\beta})^{\dagger}|\Omega\rangle$, by antisymmetry, only contains the singlet.

N>1 SUSY, with central charges

In the presence of central charges $\overline{Q}_{\dot{\alpha}A}$, Q^{A}_{α} cannot be interpreted in terms of creation/annihilation operators without rediagonalizing the basis. Recall

$$\begin{aligned} \{Q^{A}_{\alpha}, Q^{B}_{\beta}\} &= \epsilon_{\alpha\beta}X^{AB} \quad , \\ \{\overline{Q}_{\dot{\alpha}A}, \overline{Q}_{\dot{\beta}B}\} &= -\epsilon_{\dot{\alpha}\dot{\beta}}X^{*}_{AB} \quad , \end{aligned} \tag{34}$$

where X^{AB} is antisymmetric and, following Wess and Bagger, we impose the convention $X^{AB} = -X_{AB}$.

Since the central charges commute with all the other generators, we can choose any convenient basis to describe them. We will use Zumino's decomposition of a general complex antisymmetric matrix:[9]

$$X^{AB} = U^{A}_{C}\tilde{X}^{CD}(U^{T})^{B}_{D} \quad , \tag{35}$$

where, for N even, $\tilde{X}^{CD}$ has the form

$$\begin{pmatrix} (Z_1\epsilon^{ab}) & 0 & \dots & 0 \\ 0 & (Z_2\epsilon^{ab}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (Z_{\frac{N}{2}}\epsilon^{ab}) \end{pmatrix} \quad , \tag{36}$$

where $\epsilon^{ab}=i\sigma^2$. For N odd, there is an extra right-hand column of zeroes and bottom row of zeroes. In this decomposition the "eigenvalues" $Z_1, Z_2, \ldots Z_{[\frac{N}{2}]}$ are real and nonnegative.

Consider now the massive states in the rest frame. In the basis defined by Zumino's decomposition we have:

$$
\begin{aligned}
\{Q_\alpha, \overline{Q}_{\dot\beta}\} &= 2m\sigma^0_{\alpha\dot\beta}\delta^a_b\delta^L_M \quad , \\
\{Q^{aL}_\alpha, Q^{bM}_\beta\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LM} Z_M \quad , \\
\{\overline{Q}_{\dot\alpha aL}, \overline{Q}_{\dot\beta bM}\} &= -\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab}\delta_{LM} Z_M \quad ,
\end{aligned}
\tag{37}
$$

where the internal indices A, B have now been replaced by the index pairs (a, L), (b, M), with $a, b = 1, 2$ and $L, M = 1, 2, \ldots [\frac{N}{2}]$. Here and in the following, the repeated M index is **not** summed over.

It is now apparent that there are 2N pairs of creation/annihilation operators:

$$
\begin{aligned}
a^L_\alpha &= \frac{1}{\sqrt{2}}\left[Q^{1L}_\alpha + \epsilon_{\alpha\beta}\overline{Q}_{\dot\gamma 2L}\bar\sigma^{0\dot\gamma\beta}\right] \quad , \\
(a^L_\alpha)^\dagger &= \frac{1}{\sqrt{2}}\left[\overline{Q}_{\dot\alpha 1L} + \epsilon_{\dot\alpha\dot\beta}\bar\sigma^{0\dot\beta\gamma}Q^{2L}_\gamma\right] \quad , \\
b^L_\alpha &= \frac{1}{\sqrt{2}}\left[Q^{1L}_\alpha - \epsilon_{\alpha\beta}\overline{Q}_{\dot\gamma 2L}\bar\sigma^{0\dot\gamma\beta}\right] \quad , \\
(b^L_\alpha)^\dagger &= \frac{1}{\sqrt{2}}\left[\overline{Q}_{\dot\alpha 1L} - \epsilon_{\dot\alpha\dot\beta}\bar\sigma^{0\dot\beta\gamma}Q^{2L}_\gamma\right] \quad .
\end{aligned}
\tag{38}
$$

The Lorentz index structure here looks a little strange, but the important point is that $\overline{Q}_{\dot\alpha}$ transforms the same as Q^α under **spatial** rotations. Thus $(a^L_\alpha)^\dagger$, $(b^L_\alpha)^\dagger$ create states of definite spin.

The anticommutation relation are:

$$
\begin{aligned}
\{a^L_\alpha, (a^M_\beta)^\dagger\} &= (2m + Z_M)\sigma^0_{\alpha\dot\beta}\delta^L_M \quad , \\
\{b^L_\alpha, (b^M_\beta)^\dagger\} &= (2m - Z_M)\sigma^0_{\alpha\dot\beta}\delta^L_M \quad .
\end{aligned}
\tag{39}
$$

This is easily verified from Eqs 37, 38 using the relations:

$$
\begin{aligned}
\epsilon_{\alpha\delta}\bar\sigma^{0\dot\gamma\delta}\epsilon_{\dot\gamma\dot\beta} &= -\sigma^0_{\alpha\dot\beta} \quad , \\
\epsilon_{\dot\beta\dot\delta}\bar\sigma^{0\dot\delta\gamma}\epsilon_{\alpha\gamma} &= \sigma^0_{\alpha\dot\beta} \quad .
\end{aligned}
\tag{40}
$$

BPS-saturated states

Since $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ are positive definite operators, and since the Z_M are nonnegative, we deduce the following:

- For all Z_M in any SUSY irrep:

$$Z_M \leq 2m \qquad (41)$$

- When $Z_M < 2m$ the multiplicities of the massive irreps are the same as for the case of no central charges.

- The special case is when we saturate the bound, i.e. $Z_M = 2m$ for some or all Z_M. If e.g. all the Z_M saturate the bound, then all of the $(b_\alpha^L)^\dagger$ are projections onto zero norm states; thus effectively we lose half of the creation operators. This implies that this massive SUSY irrep has only $2^N(2j+1)$ states instead of $2^{2N}(2j+1)$ states.

These reduced multiplicity massive multiplets are often called **short multiplets**. The states are often referred to as **BPS-saturated states**, because of the connection to BPS monopoles in supersymmetric gauge theories. [10]

For example, let us compare the fundamental N=2 massive irreps. For N=2 there is only one central charge, Z. For $Z < 2m$ we have the **long multiplet** already discussed:

Ω_0^{long} :				
	spin	0	$\frac{1}{2}$	1
	no. of spin irreps	5	4	1
	total no. of states	5	8	3

There are a grand total of 16 states.

For $Z = 2m$ we have BPS-saturated states in a short multiplet:

Ω_0^{short} :				
	spin	0	$\frac{1}{2}$	1
	no. of spin irreps	2	1	0
	total no. of states	2	2	0

There are a grand total of 4 states. Note that the spins and number of states of this BPS-saturated massive multiplet match those of the N=2 **massless** hypermultiplet in Table 1.

Let us also compare the $j=\frac{1}{2}$ N=2 massive irreps. For $Z < 2m$ we have a long multiplet with 32 states:

$\Omega^{\text{long}}_{\frac{1}{2}}$:	spin	0	$\frac{1}{2}$	1	$\frac{3}{2}$
	no. of spin irreps	4	6	4	1
	total no. of states	4	12	12	4

For $Z = 2m$ we have a short multiplet with 8 states:

$\Omega^{\text{short}}_{\frac{1}{2}}$:	spin	0	$\frac{1}{2}$	1	$\frac{3}{2}$
	no. of spin irreps	1	2	1	0
	total no. of states	1	4	3	0

Note that the spins and number of states of this BPS-saturated massive multiplet match those of the N=2 massless vector multiplet (allowing for the fact that a massless vector eats a scalar in becoming massive).

Automorphisms of the supersymmetry algebra

In the absence of central charges, the general 4-dimensional SUSY algebra has an obvious $U(N)$ automorphism symmetry:

$$Q^A_\alpha \to U^A{}_B Q^B_\alpha, \qquad \overline{Q}_{\dot\alpha A} \to \overline{Q}_{\dot\alpha B} U^{\dagger B}{}_A \quad , \tag{42}$$

where $U^A{}_B$ is a unitary matrix. SUSY irreps on asymptotic single particle states will automatically carry a representation of the automorphism group. For massless irreps $U(N)$ is the largest automorphism symmetry which respects helicity.

For massive irreps, we have already noted that Q^α and $\overline{Q}_{\dot\alpha}$ transform the same way under spatial rotations. Assembling these into a 2N component object, one finds that the largest automorphism group which respects spin is $USp(2N)$, the unitary symplectic group of rank N. [11] In the presence of central charges, the automorphism group is still $USp(2N)$ provided that none of the central charges saturates the BPS bound: this follows from our ability to make the basis change Eqs. 38, 39. When one central charge saturates the BPS bound, the automorphism group is reduced to $USp(N)$ for N even, $USp(N+1)$ for N odd.

The automorphism symmetries give us constraints on the internal symmetry group generated by the B_ℓ. In the case of no central charges $U(N)$ is the largest possible internal symmetry group which can act nontrivially on the Q's. With a single central charge, the intertwining relation Eq. 264 implies that $USp(N)$ is the largest such group.

Supersymmetry represented on quantum fields

So far we have only discussed representations of SUSY on asymptotic states, not on quantum fields. Q_α, $\overline{Q}_{\dot\alpha}$ can be represented as superspace differential operators acting on fields. The Clifford vacuum condition Eq. 23 becomes a commutation condition:

$$[\overline{Q}_{\dot\alpha}, \Omega(x)] = 0 \quad . \tag{43}$$

For **on-shell** fields, the construction of SUSY irreps proceeds as before, with the following exception. If $\Omega(x)$ is a **real** scalar field, then the adjoint of Eq. 43 is

$$[Q_\alpha, \Omega(x)] = 0 \quad . \tag{44}$$

In that case, Eqs. 43, 44 together with the Jacobi identity for $\{[\Omega(x), Q], \overline{Q}\}$ implies that $\Omega(x)$ is a constant.

Thus we conclude that $\Omega(x)$ must be a **complex** scalar field. This has the effect that some SUSY on-shell irreps on fields have twice as many field components as the corresponding irreps for on-shell states. Because we already paired up most SUSY irreps on states to get CPT eigenstates, this doubling really only effects the SUSY irreps based on the special case Ω_λ, $\lambda = -N/4$. The first example is the massless N=2 hypermultiplet. On asymptotic single particle states this irrep consists of 4 states (see Table 1); the massless N=2 hypermultiplet on fields, however, has 8 real components.

2.5 N=1 rigid superspace

Relativistic quantum field theory relies upon the fact that the spacetime coordinates x^m parametrize the coset space defined as the Poincaré group modded out by the Lorentz group. Clearly it is desirable to find a similar coordinatization for supersymmetric field theory. For simplicity we will discuss the case of N=1 SUSY, deferring N>1 SUSY until Section 5.

The first step is to rewrite the N=1 SUSY algebra as a Lie algebra. This requires that we introduce constant Grassmann spinors θ^α, $\overline{\theta}_{\dot\alpha}$:

$$\{\theta^\alpha, \theta^\beta\} = \{\overline{\theta}_{\dot\alpha}, \overline{\theta}_{\dot\beta}\} = \{\theta^\alpha, \overline{\theta}_{\dot\beta}\} = 0 \quad . \tag{45}$$

This allows us to replace the anticommutators in the N=1 SUSY algebra with commutators:

$$[\theta Q, \overline{\theta}\,\overline{Q}] \;=\; 2\theta\sigma^m\overline{\theta}P_m \quad ,$$

$$
\begin{aligned}
[\theta Q, \theta Q] &= 0 \quad , \\
[\overline{\theta}\,\overline{Q}, \overline{\theta}\,\overline{Q}] &= 0 \quad .
\end{aligned} \tag{46}
$$

Note we have now begun to employ the spinor summation convention discussed in the Appendix.

Given a Lie algebra we can exponentiate to get the general group element:

$$
G(x, \theta, \overline{\theta}, \omega) = e^{i[-x^m P_m + \theta Q + \overline{\theta}\,\overline{Q}]} e^{-\frac{1}{2}\omega^{mn} M_{mn}} \quad , \tag{47}
$$

where the minus sign in front of x^m is a convention. Note that this form of the general N=1 superPoincaré group element is unitary since $(\theta Q)^\dagger = \overline{\theta}\,\overline{Q}$.

From Eq. 47 it is clear that $(x^m, \theta^\alpha, \overline{\theta}_{\dot{\alpha}})$ parametrizes a 4+4 dimensional coset space: N=1 superPoincaré **mod** Lorentz. This coset space is more commonly known as **N=1 rigid superspace**: "rigid" refers to the fact that we are discussing **global** supersymmetry.

There are great advantages to constructing supersymmetric field theories in the superspace/superfield formalism, just as there are great advantages to constructing relativistic quantum field theories in a manifestly Lorentz covariant formalism. Our rather long technical detour into superspace and superfield constructions will pay off nicely when we begin the construction of supersymmetric actions.

Superspace derivatives

Here we collect the basic notation and properties of N=1 superspace derivatives.

$$
\begin{aligned}
\partial_\alpha &= \frac{\partial}{\partial\theta^\alpha} \quad , \quad \partial^\alpha = \frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\partial_\beta \quad , \\
\overline{\partial}^{\dot{\alpha}} &= \frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}} \quad , \quad \overline{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\partial}^{\dot{\beta}} \quad , \\
\partial_\alpha\theta^\beta &= \delta_\alpha^\beta \quad , \quad \overline{\partial}^{\dot{\alpha}}\overline{\theta}_{\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} \quad , \\
\partial^\alpha\theta^\beta &= -\epsilon^{\alpha\beta} \quad , \quad \partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta} \quad , \\
\overline{\partial}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} &= -\epsilon^{\dot{\alpha}\dot{\beta}} \quad , \quad \overline{\partial}_{\dot{\alpha}}\overline{\theta}_{\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \quad , \\
\partial_\alpha\theta^\beta\theta^\gamma &= \delta_\alpha^\beta\theta^\gamma - \delta_\alpha^\gamma\theta^\beta \quad , \\
\partial_\alpha(\theta\theta) &= 2\theta_\alpha \quad , \quad \overline{\partial}_{\dot{\alpha}}(\overline{\theta\theta}) = -2\overline{\theta}_{\dot{\alpha}} \quad , \\
\partial^2(\theta\theta) &= 4 \quad , \quad \overline{\partial}^2(\overline{\theta\theta}) = 4 \quad .
\end{aligned} \tag{48}
$$

Superspace integration

We begin with the Berezin integral for a single Grassmann parameter θ:

$$\begin{aligned} \int d\theta\, \theta &= 1 \quad, \\ \int d\theta &= 0 \quad, \\ \int d\theta\, f(\theta) &= f_1 \quad, \end{aligned} \tag{49}$$

where we have used the fact that an arbitrary function of a single Grassmann parameter θ has the Taylor series expansion $f(\theta) = f_0 + \theta f_1$.

We note three facts which follow from the definitions of Eq. 49.

- Berezin integration is translationally invariant:

$$\begin{aligned} \int d(\theta+\xi)\, f(\theta+\xi) &= \int d\theta\, f(\theta) \quad, \\ \int d\theta\, \frac{d}{d\theta} f(\theta) &= 0 \quad. \end{aligned} \tag{50}$$

- Berezin integration is equivalent to differentiation:

$$\frac{d}{d\theta} f(\theta) = f_1 = \int d\theta\, f(\theta) \quad. \tag{51}$$

- We can define a Grassmann delta function by

$$\delta(\theta) \equiv \theta \quad. \tag{52}$$

These results are easily generalized to the case of the $N=1$ superspace coordinates θ^α, $\bar{\theta}_{\dot{\alpha}}$. The important notational conventions are:

$$\begin{aligned} d^2\theta &= -\frac{1}{4} d\theta^\alpha\, d\theta^\beta\, \epsilon_{\alpha\beta} \quad, \\ d^2\bar{\theta} &= -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}}\, d\bar{\theta}_{\dot{\beta}}\, \epsilon^{\dot{\alpha}\dot{\beta}} \quad, \\ d^4\theta &= d^2\theta\, d^2\bar{\theta} \quad. \end{aligned} \tag{53}$$

Using this notation and the spinor summation convention, we have the following identities:

$$\begin{aligned} \int d^2\theta\, \theta\theta &= 1 \quad, \\ \int d^2\bar{\theta}\, \bar{\theta}\bar{\theta} &= 1 \quad. \end{aligned} \tag{54}$$

Superspace covariant derivatives

If we wanted to treat a general curved N=1 superspace, we would have to introduce a 4+4=8-dimensional vielbein and spin connection. Using M to denote an 8-dimensional superspace index, and A to denote an 8-dimensional super-tangent space index, we can write the vielbein and spin connection as E_M^A and W_A^{mn} respectively. The general form of a covariant derivative in such a space is thus

$$D_M = E_M^A(\partial_A + \frac{1}{2}W_A^{mn}M_{mn}) \quad , \tag{55}$$

where $\partial_A = (\partial_m, \partial_\alpha, \overline{\partial}_{\dot\alpha})$.

Naively one might expect that D_M reduces to ∂_M for N=1 rigid superspace, since the rigid superspace has zero curvature. However it is possible to show[3] that N=1 rigid superspace has nonzero **torsion**, and thus that the vielbein is nontrivial. The covariant derivatives for N=1 rigid superspace are given by:

$$\begin{aligned}
D_m &= \partial_m \quad , \\
D_\alpha &= \partial_\alpha + i\sigma^m_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_m \quad , \\
\overline{D}_{\dot\alpha} &= -\overline{\partial}_{\dot\alpha} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m \quad , \\
D^\alpha &= -\partial^\alpha - i\overline{\theta}_{\dot\beta}\overline{\sigma}^{m\dot\beta\alpha}\partial_m \quad , \\
\overline{D}^{\dot\alpha} &= \overline{\partial}^{\dot\alpha} + i\overline{\sigma}^{m}_{\dot\alpha\beta}\theta_\beta\partial_m \quad .
\end{aligned} \tag{56}$$

3 N=1 Superfields

3.1 *The general N=1 scalar superfield*

The general scalar superfield $\Phi(x,\theta,\overline{\theta})$ is just a scalar function in N=1 rigid superspace. It has a finite Taylor expansion in powers of θ^α, $\overline{\theta}_{\dot\alpha}$; this is known as the **component expansion** of the superfield:

$$\begin{aligned}
\Phi(x,\theta,\overline{\theta}) = f(x) + \theta\phi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) \\
+\theta\sigma^m\overline{\theta}v_m(x) + (\theta\theta)\overline{\theta}\overline{\lambda}(x) + (\overline{\theta\theta})\theta\psi(x) + (\theta\theta)(\overline{\theta\theta})d(x) \quad .
\end{aligned} \tag{57}$$

The component fields in Eq 57 are complex: redundant terms like $\overline{\theta}\overline{\sigma}^m\theta v_m$ have already been removed using the Fierz identities listed in the Appendix. The fermionic component fields $\phi(x)$, $\overline{\chi}(x)$, $\overline{\lambda}(x)$, and $\psi(x)$ are Grassmann odd, i.e. they anticommute with each other and with θ, $\overline{\theta}$.

To compute the effect of an infinitesimal N=1 SUSY transformation on a general scalar superfield, we need the explicit representation of Q, $\overline{Q}$ as superspace differential operators. Recall that for ordinary scalar fields the translation generator P_m is represented (with our conventions) by the differential operator $i\partial_m$. Let ξ^α be a constant Grassmann complex Weyl spinor, and consider the effect of left multiplication by a "supertranslation" generator $G(y,\xi)$ on an arbitrary coset element $\Omega(x,\theta,\bar\theta)$:

$$
\begin{aligned}
G(y,\xi)\Omega(x,\theta,\bar\theta) &= e^{i[-y^m P_m+\xi Q+\overline{\xi Q}]}e^{i[-x^m P_m+\theta Q+\overline{\theta Q}]} \\
&= e^{i[-(x^m+y^m)P_m+(\theta^\alpha+\xi^\alpha)Q_\alpha+(\bar\theta_{\dot\alpha}+\bar\xi_{\dot\alpha})\overline{Q}^{\dot\alpha}+\frac{1}{2}[\xi Q,\overline{\theta Q}]+\frac{1}{2}[\xi\overline{Q},\theta Q]} \\
&= \Omega\left((x^m-y^m-i\xi\sigma^m\bar\theta+i\theta\sigma^m\bar\xi),\theta+\xi,\bar\theta+\bar\xi\right) \quad ,
\end{aligned}
\tag{58}
$$

where, to obtain the last expression, we have used the commutators:

$$
\begin{aligned}
[\xi Q,\bar\theta\overline{Q}] &= 2\xi\sigma^m\bar\theta P_m \quad , \\
[\xi\overline{Q},\theta Q] &= -2\theta\sigma^m\bar\xi P_m \quad .
\end{aligned}
\tag{59}
$$

From Eq. 58 we see that, with our conventions, P_m, Q, and $\overline{Q}$ have the following representation as superspace differential operators:

$$
\begin{aligned}
P_m &: i\partial_m \quad , \\
Q_\alpha &: \partial_\alpha - i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m \quad , \\
\overline{Q}_{\dot\alpha} &: \bar\partial_{\dot\alpha} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m \quad .
\end{aligned}
\tag{60}
$$

It is now a trivial matter to compute the infinitesimal variation of the general scalar superfield Eq.57 under an N=1 SUSY transformation:

$$
\begin{aligned}
\delta_\xi\Phi(x,\theta,\bar\theta) &= (\xi Q+\bar\xi\overline{Q})\Phi(x,\theta,\bar\theta) \\
&= \xi\phi+\bar\xi\bar\chi+i\theta\sigma^m\bar\xi\partial_m f+2\xi\theta m+\theta\sigma^m\bar\xi v_m-i\xi\sigma^m\bar\theta\partial_m f \\
&+2\bar\xi\bar\theta n+\xi\sigma^m\bar\theta v_m+i(\theta\sigma^m\bar\xi)\theta\partial_m\phi+(\theta\theta)(\bar\xi\bar\lambda)-i(\xi\sigma^m\bar\theta)\bar\theta\partial_m\bar\chi \\
&+(\overline{\theta\theta})(\xi\psi)-i\xi\sigma^m\bar\theta\theta\partial_m\phi+i\theta\sigma^m\bar\xi\bar\theta\partial_m\bar\chi+2(\xi\theta)(\bar\theta\bar\lambda)+2(\bar\xi\bar\theta)(\theta\psi) \\
&-i\xi\sigma^m\bar\theta(\theta\theta)\partial_m m+i\theta\sigma^m\bar\xi\theta\sigma^n\bar\theta\partial_m v_n+2(\theta\theta)(\bar\xi\bar\theta)d+i\theta\sigma^m\bar\xi(\overline{\theta\theta})\partial_m n \\
&-i\xi\sigma^m\bar\theta\theta\sigma^n\bar\theta\partial_m v_n+2(\xi\theta)(\overline{\theta\theta})d-i\xi\sigma^m\bar\theta(\theta\theta)\bar\theta\partial_m\bar\lambda+i\theta\sigma^m\bar\xi(\overline{\theta\theta})\theta\partial_m\psi \quad .
\end{aligned}
\tag{61}
$$

Using the Fierz identities, we then have that the component fields of Φ transform as follows:

$$
\begin{aligned}
\delta_\xi f &= \xi\phi+\bar\xi\bar\chi \quad , \\
\delta_\xi\phi_\alpha &= 2\xi_\alpha m+\sigma^m_{\alpha\dot\beta}\bar\xi^{\dot\beta}[i\partial_m f+v_m] \quad , \\
\delta_\xi\bar\chi^{\dot\alpha} &= 2\bar\xi^{\dot\alpha}n+\xi^\beta\sigma^m_{\beta\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}[i\partial_m f-v_m] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_\xi m &= \bar{\xi}\bar{\lambda} - \frac{i}{2}\partial_m\phi\sigma^m\bar{\xi} \quad , \\
\delta_\xi n &= \xi\psi + \frac{i}{2}\xi\sigma^m\partial_m\bar{\chi} \quad , \\
\delta_\xi v_m &= \xi\sigma_m\bar{\lambda} + \psi\sigma_m\bar{\xi} + \frac{i}{2}\xi\partial_m\phi - \frac{i}{2}\partial_m\bar{\chi}\bar{\xi} \quad , \\
\delta_\xi \bar{\lambda}^{\dot{\alpha}} &= 2\bar{\xi}^{\dot{\alpha}} d + \frac{i}{2}\bar{\xi}^{\dot{\alpha}}\partial^m v_m + i(\xi\sigma^m\epsilon)^{\dot{\alpha}}\partial_m m \quad ; \\
\delta_\xi \psi_\alpha &= 2\xi_\alpha d - \frac{i}{2}\xi_\alpha\partial^m v_m + i(\sigma^m\bar{\xi})_\alpha\partial_m n \quad , \\
\delta_\xi d &= \frac{i}{2}\partial_m\left[\psi\sigma^m\bar{\xi} + \xi\sigma^m\bar{\lambda}\right] \quad .
\end{aligned}
\tag{62}
$$

Note the important fact that the complex scalar component field $d(x)$ transforms by a total derivative.

We have thus demonstrated that the general scalar superfield forms a basis for an (off-shell) linear representation of N=1 supersymmetry. However this representation is **reducible.** To see this, suppose we impose the following constraints on the component fields of Φ:

$$
\begin{aligned}
\chi(x) &= 0 \quad , \\
n(x) &= 0 \quad , \\
v_m(x) &= i\partial_m f(x) \quad , \\
\bar{\lambda}(x) &= \frac{i}{2}\partial_m\phi\sigma^m \quad , \\
\psi(x) &= 0 \quad , \\
d(x) &= -\frac{1}{4}\Box f(x) \quad .
\end{aligned}
\tag{63}
$$

It is easy to verify that the N=1 SUSY component field transformations Eq. 62 respect these constraints. Thus the constrained superfield by itself defines an off-shell linear representation of N=1 SUSY (in fact, an irreducible representation). This suffices to prove that representation defined by Φ is reducible. In fact there are several ways of extracting irreps by constraining Φ, however the general scalar superfield is **not fully reducible.** i.e. the reducible representation is not a direct sum of irreducible representations.

We can also use Φ to demonstrate the importance of the superspace covariant derivatives D_α. $\overline{D}_{\dot{\alpha}}$. Consider $\overline{\partial}_{\dot{\alpha}}\Phi(x,\theta,\overline{\theta})$: this has fewer component fields than Φ since. for example. there is no $(\theta\theta)(\overline{\theta\theta})$ term in its component expansion. However the commutator of $\overline{\partial}_{\dot{\alpha}}$ with ξQ is nonvanishing:

$$
[\,\overline{\partial}_{\dot{\alpha}}, \xi Q\,] = i\xi^\beta\sigma^m_{\beta\dot{\alpha}}\partial_m \quad , \tag{64}
$$

and this implies that an N=1 SUSY transformation generates a $(\theta\theta)(\overline{\theta\theta})$ term. Thus $\overline{\partial}_{\dot\alpha}\Phi(x,\theta,\overline{\theta})$ is not a true superfield in the sense of providing a basis for a linear representation of supersymmetry.

The superspace covariant derivatives, on the other hand, anticommute with Q and $\overline{Q}$:

$$\begin{aligned}\{D_\alpha, Q_\beta\} &= \{D_\alpha, \overline{Q}_{\dot\beta}\} &= 0 \quad , \\ \{\overline{D}_{\dot\alpha}, Q_\beta\} &= \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} &= 0 \quad ,\end{aligned} \tag{65}$$

Thus if Φ is a general scalar superfield, then $\partial_m\Phi$, $D_\alpha\Phi$, and $\overline{D}_{\dot\alpha}\Phi$ are also superfields.

3.2 N=1 chiral superfields

An N=1 chiral superfield is obtained by the constraints Eq. 63 imposed on a general scalar superfield. A more elegant and useful definition comes from realizing that Eq. 63 is equivalent to the following **covariant constraint**:

$$\overline{D}_{\dot\alpha}\Phi = 0 \quad . \tag{66}$$

Covariant constraints are constraints which involve only superfields (and covariant derivatives of superfields, since these are also superfields). It is a plausible but nonobvious fact that the superfields which define irreducible off-shell linear representations of supersymmetry can always be obtained by imposing covariant constraints on unconstrained superfields.

Let us find the most general solution to the covariant constraint Eq. 66. Define new bosonic coordinates y^m in N=1 rigid superspace:

$$y^m = x^m + i\theta\sigma^m\overline{\theta} \quad . \tag{67}$$

We note in passing that the funny minus sign convention in Eq. 47 is tied the fact that sign in Eq. 67 above is plus. Since

$$\begin{aligned}\overline{D}_{\dot\alpha}\, y^m &= 0 \quad , \\ \overline{D}_{\dot\alpha}\, \theta^\alpha &= 0 \quad ,\end{aligned} \tag{68}$$

it is clear that any function $\Phi(y,\theta)$ of y^m and θ^α (but not $\overline{\theta}_{\dot\alpha}$) satisfies

$$\overline{D}_{\dot\alpha}\, \Phi(y,\theta) = 0 \quad . \tag{69}$$

It is easy to see that, since $\overline{D}_{\dot\alpha}$ obeys the chain rule, this is not just a particular solution of Eq. 66 but is in fact the most general solution.

Thus we may write the most general N=1 chiral superfield as:

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \quad , \tag{70}$$

where $A(y)$, $F(y)$ are complex scalar fields, while $\psi^\alpha(y)$ is a complex left-handed Weyl spinor. The $\sqrt{2}$ is a convention. There are $4+4=8$ real off-shell field components: this is twice the number in the on-shell fundamental N=1 massive irrep.

The full θ, $\bar{\theta}$ component expansion is obtained by using the Fierz identity Eq. 247. The result is:

$$\begin{aligned}\Phi(y,\theta) &= A(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\ &+ i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{i}{\sqrt{2}}(\theta\theta)\partial_m\psi(x)\sigma^m\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A(x) \quad .\end{aligned} \tag{71}$$

An infinitesimal N=1 SUSY transformation on the chiral superfield yields:

$$\begin{aligned}\delta A &= \sqrt{2}\xi\psi \quad , \\ \delta\psi &= \sqrt{2}\xi F + \sqrt{2}i\sigma^m\bar{\xi}\partial_m A \quad , \\ \delta F &= -\sqrt{2}i\partial_m\psi\sigma^m\bar{\xi} \quad .\end{aligned} \tag{72}$$

Note that $\delta F(x)$ is a total derivative.

Antichiral superfields, i.e. right-handed chiral superfields, are defined in the obvious way. In particular, if $\Phi(y,\theta)$ is a chiral superfield, then $\Phi^\dagger$ is an antichiral superfield; it satisfies

$$\begin{aligned}D_\alpha\Phi^\dagger &= 0 \quad , \\ \Phi^\dagger &= \Phi^\dagger(y^\dagger,\bar{\theta}); \qquad y^\dagger = x^m - i\theta\sigma^m\bar{\theta} \quad .\end{aligned} \tag{73}$$

Since D_α and $\overline{D}_{\dot{\alpha}}$ obey the chain rule, any product of chiral superfields is also a chiral superfield, while any product of antichiral superfields is also an antichiral superfield. However it is also clear that if $\Phi(y,\theta)$ is a chiral superfield, the following are **not** chiral superfields:

$$\begin{aligned}&\Phi^\dagger\Phi \quad , \\ &\Phi + \Phi^\dagger \quad .\end{aligned}$$

For future reference, let us write down the expressions for the covariant derivatives acting on functions of $(y,\theta,\bar{\theta})$:

$$
\begin{aligned}
D_\alpha &= \partial_\alpha + 2i\sigma^m_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_m \quad , \\
\overline{D}_{\dot\alpha} &= -\overline{\partial}_{\dot\alpha} \quad , \\
D^\alpha &= -\partial^\alpha - 2i\overline{\theta}_{\dot\beta}\bar\sigma^{m\dot\beta\alpha}\partial_m \quad , \\
\overline{D}^{\dot\alpha} &= \overline{\partial}^{\dot\alpha} \quad ,
\end{aligned}
\tag{74}
$$

where of course here ∂_m is a partial derivative with respect to y^m rather than x^m.

3.3 N=1 vector superfields

Vector superfields are defined from the general scalar superfield by imposing a covariant reality constraint:

$$
V(x,\theta,\overline{\theta}) = V^\dagger(x,\theta,\overline{\theta}) \quad , \tag{75}
$$

or, in components:

$$
\begin{aligned}
f &= f^* \quad , \\
\bar\chi &= \phi^* \quad , \\
m &= n^* \quad , \\
v_m &= v_m^* \quad , \\
\bar\lambda &= \psi^* \quad , \\
d &= d^* \quad .
\end{aligned}
\tag{76}
$$

Thus in components we have 4 real scalars, 2 complex Weyl spinors (equivalently, 2 Majorana spinors), and 1 real vector. The 8+8 = 16 real components in this off-shell irrep are twice the number in the on-shell $\Omega_{\frac{1}{2}}$ massive irrep.

The presence of a real vector field in the N=1 vector multiplet suggests we use vector superfields to construct supersymmetric gauge theories. But first we must deduce the superfield generalization of gauge transformations.

Wess-Zumino gauge

If $\Phi(y,\theta)$ is a chiral superfield, then $\Phi + \Phi^\dagger$ is a special case of a vector superfield. In components:

$$
\begin{aligned}
\Phi + \Phi^\dagger = {}& (A + A^*) + \sqrt{2}\theta\psi + \sqrt{2}\overline{\theta}\bar\psi + \theta\theta F + \overline{\theta}\overline{\theta}F^* + i\theta\sigma^m\overline{\theta}\partial_m(A - A^*) \\
&+ \frac{i}{\sqrt{2}}(\theta\theta)\overline{\theta}\bar\sigma^m\partial_m\psi + \frac{i}{\sqrt{2}}(\overline{\theta}\overline{\theta})\theta\sigma^m\partial_m\bar\psi - \frac{1}{4}(\theta\theta)(\overline{\theta}\overline{\theta})\Box(A + A^*) \quad .
\end{aligned}
\tag{77}
$$

From this we see that we can define the superfield analog of an infinitesimal abelian gauge transformation to be

$$V \rightarrow V + \Phi + \Phi^{\dagger} \quad , \tag{78}$$

since this definition gives the correct infinitesimal transformation for the vector component:

$$\begin{aligned} v_m &\rightarrow v_m + \partial_m \Lambda \quad ; \\ \Lambda &= i(A - A^*) \quad . \end{aligned} \tag{79}$$

The meaning of the "bigger" superfield transformation Eq. 78 is that any superfield action invariant under abelian gauge transformations will also be independent of several component fields of $V(x, \theta, \bar{\theta})$. More precisely, notice that the first 5 component fields of $\Phi + \Phi^{\dagger}$ in Eq. 77 are completely unconstrained. This means that without loss of generality we can decompose any vector superfield as follows:

$$V(x, \theta, \bar{\theta}) = V_{WZ} + \Phi + \Phi^{\dagger} \quad , \tag{80}$$

where V_{WZ} only has 4 component fields instead of 9:

$$V_{WZ} = -\theta\sigma^m\bar{\theta}v_m + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\overline{\theta\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})D \quad , \tag{81}$$

where, to conform with Wess and Bagger, I have changed notation slightly:

$$\begin{aligned} v_m &\rightarrow -v_m \quad , \\ \bar{\lambda} &\rightarrow i\bar{\lambda} \quad , \\ d &\rightarrow \frac{1}{2}D \quad . \end{aligned}$$

V_{WZ} is known as the **Wess-Zumino gauge-fixed superfield.**

This decomposition is unambiguous **except** for the remaining freedom to shift part of v_m into the corresponding component of $\Phi + \Phi^{\dagger}$, i.e. $v_m \rightarrow v_m - i\partial_m(A - A^*)$. Thus fixing Wess-Zumino gauge **does not** fix the abelian gauge freedom.

3.4 The supersymmetric field strength

Note that the supersymmetry transformations do not respect the Wess-Zumino gauge-fixing decomposition. This is somewhat disappointing since it means that a superfield formulation in terms of $V(x, \theta, \bar{\theta})$ necessarily carries around a number of superfluous fields. We can however define a different superfield which has the property that it only contains the Wess-Zumino gauge-fixed component fields $v_m(x)$, $\lambda(x)$, and $D(x)$.

We define left and right-handed **spinor superfields** W_α, $\overline{W}_{\dot\alpha} = (W_\alpha)^\dagger$:

$$\begin{aligned} W_\alpha &= -\frac{1}{4}(\overline{D}\overline{D})D_\alpha V(x,\theta,\bar\theta) \quad , \\ \overline{W}_{\dot\alpha} &= -\frac{1}{4}(DD)\overline{D}_{\dot\alpha} V(x,\theta,\bar\theta) \quad . \end{aligned} \tag{82}$$

An equivalent definition, which we will need when we go from the abelian to the nonabelian case, is:

$$\begin{aligned} W_\alpha &= -\frac{1}{8}(\overline{D}\overline{D})\mathrm{e}^{-2V} D_\alpha \mathrm{e}^{2V} \quad , \\ \overline{W}_{\dot\alpha} &= \frac{1}{8}(DD)\mathrm{e}^{2V}\overline{D}_{\dot\alpha}\mathrm{e}^{-2V} \quad . \end{aligned} \tag{83}$$

W_α is a chiral superfield:

$$\overline{D}_{\dot\alpha} W_\beta = -\frac{1}{4}\overline{D}_{\dot\alpha}(\overline{D}_{\dot\gamma}\overline{D}^{\dot\gamma})D_\beta V(x,\theta,\bar\theta) = 0 \quad , \tag{84}$$

where we have used the fact that since the $\overline{D}$'s anticommute and have only 2 components, $(\overline{D})^3 = 0$.

$\overline{W}_{\dot\alpha}$ is an antichiral superfield. W_α is not a general chiral spinor superfield, because W_α and $\overline{W}_{\dot\alpha}$ are related by an additional covariant constraint:

$$\overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha} = D^\alpha W_\alpha \quad . \tag{85}$$

This constraint follows trivially from Eq. 82:

$$\begin{aligned} \overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha} = \quad & \epsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}\overline{W}_{\dot\beta} && = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}(DD)\overline{D}_{\dot\beta}V \\ = \quad & -\tfrac{1}{4}(\overline{D}\overline{D})(DD)V && = -\frac{1}{4}D^\alpha(\overline{D}\overline{D})D_\alpha V \\ = \quad & D^\alpha W_\alpha && . \end{aligned} \tag{86}$$

W_α and $\overline{W}_{\dot\alpha}$ are both invariant under the transformation Eq. 78. Let us prove this for W_α:

$$\begin{aligned} W_\alpha \quad &\to \quad -\frac{1}{4}(\overline{D}\overline{D})D_\alpha(V+\Phi+\Phi^\dagger) \quad , \\ &= \quad W_\alpha - \frac{1}{4}(\overline{D}\overline{D})D_\alpha\Phi \quad , \quad (\text{since } D_\alpha\Phi^\dagger = 0) \\ &= \quad W_\alpha + \frac{1}{4}\overline{D}^{\dot\beta}\{\overline{D}_{\dot\beta}, D_\alpha\}\Phi \quad , \quad (\text{since } \overline{D}_{\dot\beta}\Phi = 0) \\ &= \quad W_\alpha \quad , \end{aligned} \tag{87}$$

where in the last step we have used:

$$\begin{aligned}
\{\overline{D}_{\dot\beta}, D_\alpha\} &= -2\sigma^m_{\alpha\dot\beta} P_m \quad , \\
[\overline{D}^{\dot\beta}, P_m] &= 0 \quad .
\end{aligned} \tag{88}$$

Since W_α and $\overline{W}_{\dot\alpha}$ are both invariant under Eq. 78. there is no loss of generality in computing their components in Wess-Zumino gauge, i.e. write

$$\begin{aligned}
W_\alpha &= -\frac{1}{4}(\overline{D}\overline{D})D_\alpha V_{WZ}(x,\theta,\overline{\theta}) \quad , \\
\overline{W}_{\dot\alpha} &= -\frac{1}{4}(DD)\overline{D}_{\dot\alpha} V_{WZ}(x,\theta,\overline{\theta}) \quad .
\end{aligned} \tag{89}$$

Since $W_\alpha = W_\alpha(y,\theta)$ we write

$$V_{WZ}(x,\theta,\overline{\theta}) = V_{WZ}(y - i\theta\sigma\overline{\theta},\theta,\overline{\theta}) \tag{90}$$

and expand W_α in component fields which are functions of y:

$$\begin{aligned}
W_\alpha &= -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n\theta)_\alpha(\partial_m v_n - \partial_n v_m)(y) \\
&\quad +(\theta\theta)\sigma^m_{\alpha\dot\beta}\partial_m\bar\lambda^{\dot\beta}(y) \quad , \\
\overline{W}_{\dot\alpha} &= i\bar\lambda_{\dot\alpha}(y^\dagger) + \overline{\theta}_{\dot\alpha}D(y^\dagger) + \frac{i}{2}(\bar\sigma^m\sigma^n\overline{\theta})_{\dot\alpha}(\partial_m v_n - \partial_n v_m)(y^\dagger) \\
&\quad -(\overline{\theta}\overline{\theta})\bar\sigma^{m\beta}_{\dot\alpha}\partial_m\lambda_\beta(y^\dagger) \quad .
\end{aligned} \tag{91}$$

So indeed W_α, $\overline{W}_{\dot\alpha}$ contain only the component fields

$$\lambda,\ D,\ f_{mn} \equiv \partial_m v_n - \partial_n v_m \quad .$$

This is an irreducible off-shell multiplet known as the **curl multiplet** or **field strength multiplet**: it has 4+1+3 = 8 real components.

Nonabelian generalization

We can exponentiate the infinitesimal abelian transformation Eq. 78 to obtain the finite transformation

$$e^V \rightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \quad , \tag{92}$$

where, to conform with the standard notation of Ferrara and Zumino,[12] we now denote the chiral superfields of Eq. 78 by:

$$\begin{aligned}
\Phi &\rightarrow i\Lambda \quad , \\
\Phi^\dagger &\rightarrow -i\Lambda^\dagger \quad .
\end{aligned}$$

To obtain the nonabelian generalization we write

$$\begin{aligned} V &\rightarrow T^a_{ij} V_a \quad , \\ \Lambda &\rightarrow T^a_{ij} \Lambda_a \quad ; \\ [T^a, T^b] &= i f^{abc} T^c \quad , \\ \mathrm{tr}\, T^a T^b &= \delta^{ab} \quad , \end{aligned} \tag{93}$$

where the T^a_{ij} are the hermitian generators of some Lie algebra. The form of the nonabelian transformation is then the same as Eq. 92.

To find the infinitesimal nonabelian transformation, we can apply the Baker-Campbell-Hausdorff formula to Eq. 92. One can show that, to first order in Λ, Eq. 92 reduces to: [13]

$$\delta V = i L_{V/2}(\Lambda + \Lambda^\dagger) + i L_{V/2} \coth L_{V/2}(\Lambda - \Lambda^\dagger) \quad , \tag{94}$$

where the operation $L_X Y$ denotes the Lie derivative:

$$\begin{aligned} L_X Y &= [X, Y] \quad , \\ (L_X)^2 Y &= [X, [X, Y]] \quad , \\ &\text{etc.} \end{aligned} \tag{95}$$

Eq. 94 is meant to be evaluated by its Taylor series expansion, using

$$x \coth x = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \ldots \quad . \tag{96}$$

This becomes much more illuminating if we fix the nonabelian equivalent of Wess-Zumino gauge. Unlike the abelian case, the relationship between the component fields of $V(x, \theta, \bar{\theta})$ and $\Lambda(y, \theta)$ in the Wess-Zumino gauge fixing is nonlinear, due to the complicated form of Eq. 94. However the end result is the same: $V_{WZ}(x, \theta, \bar{\theta})$ is as given in Eq. 81. Furthermore, as in the abelian case, the Wess-Zumino decomposition does not fix the freedom to perform gauge transformations parametrized by the scalar component of $\Phi - \Phi^\dagger \equiv i(\Lambda + \Lambda^\dagger)$.

Consider then the transformation Eq. 94 with V replaced by V_{WZ}, and with only the scalar component of $\Lambda + \Lambda^\dagger$ nonvanishing (which also implies that only the $\theta\sigma\bar{\theta}$ component of $\Lambda - \Lambda^\dagger$ is nonvanishing). Clearly only the first term in the Taylor series expansion of the hyperbolic cotangent remains, since the next higher order term gives something proportional to $\theta^3 \bar{\theta}^3$. Thus having fixed Wess-Zumino gauge the infinitesimal nonabelian gauge transformation is just

$$\delta V = i(\Lambda - \Lambda^\dagger) - \frac{i}{2}[(\Lambda + \Lambda^\dagger), V] \quad . \tag{97}$$

This implies the usual nonabelian gauge transformations for the component fields $v_m(x)$, $\lambda(x)$, and $D(x)$ ($v_m(x)$ is the nonabelian gauge field while $\lambda(x)$ and $D(x)$ are matter fields in the adjoint representation).

W_α and $\overline{W}_{\dot\alpha}$ are given by Eq. 83 in the nonabelian case. Let us compute how W_α and $\overline{W}_{\dot\alpha}$ transform under Eq. 92. First notice that, under the transformation Eq. 92:

$$e^{-2V} D_\alpha e^{2V} \to e^{-i2\Lambda} e^{-2V} \left(D_\alpha e^{2V}\right) e^{i2\Lambda} + e^{-i2\Lambda} D_\alpha e^{i2\Lambda} \quad , \tag{98}$$

which follows from the fact that $D_\alpha \Lambda^\dagger = 0$. Thus, using also the fact that $\overline{D}_{\dot\alpha}$ commutes with Λ, we see that

$$W_\alpha \to e^{-i2\Lambda} W_\alpha e^{i2\Lambda} - \frac{1}{8} e^{-i2\Lambda} (\overline{D}\overline{D}) D_\alpha e^{i2\Lambda} \quad . \tag{99}$$

Furthermore the second term vanishes, just as in Eq. 87, using the identities Eq. 88. So our final result is that W_α and $\overline{W}_{\dot\alpha}$ transform covariantly in the nonabelian case:

$$\begin{aligned} W_\alpha &\to e^{-i2\Lambda} W_\alpha e^{i2\Lambda} \quad , \\ \overline{W}_{\dot\alpha} &\to e^{-i2\Lambda^\dagger} \overline{W}_{\dot\alpha} e^{i2\Lambda^\dagger} \quad . \end{aligned} \tag{100}$$

Let us be more explicit in the nonabelian case about the derivation of the component expansion for W_α. There is no loss of generality in computing this in Wess-Zumino gauge. From the definition Eq. 83 we have the explicit expression:

$$\begin{aligned} W_\alpha &= -\frac{1}{8}(\overline{D}\overline{D}) e^{-2V_{WZ}} D_\alpha e^{2V_{WZ}} \quad , \\ &= -\frac{1}{4}(\overline{D}\overline{D}) D_\alpha V_{WZ} + \frac{1}{2}(\overline{D}\overline{D}) V_{WZ} D_\alpha V_{WZ} - \frac{1}{4}(\overline{D}\overline{D}) D_\alpha V_{WZ}^2 \end{aligned} \tag{101}$$

where we have used our knowledge (see Eq. 81) of the component expansion for $V_{WZ}(y - i\theta\sigma\overline{\theta}, \theta, \overline{\theta})$:

$$\begin{aligned} V_{WZ}(y - i\theta\sigma\overline{\theta}, \theta, \overline{\theta}) &= -\theta\sigma^m\overline{\theta} v_m(y) + i(\theta\theta)\overline{\theta}\overline{\lambda}(y) - i(\overline{\theta}\overline{\theta})\theta\lambda(y) \\ &\quad + \frac{1}{2}(\theta\theta)(\overline{\theta}\overline{\theta})\left(D(y) + i\partial^m v_m(y)\right) \quad . \end{aligned} \tag{102}$$

Using the form Eq. 75 for D_α acting on functions of $(y, \theta, \overline{\theta})$, we have:

$$\begin{aligned} D_\alpha V_{WZ} = &-\sigma^m_{\alpha\dot\beta}\overline{\theta}^{\dot\beta} v_m(y) + 2i\theta_\alpha\overline{\theta}\overline{\lambda}(y) - i(\overline{\theta}\overline{\theta})\lambda_\alpha(y) \\ &+ \theta_\alpha(\overline{\theta}\overline{\theta})\left(D(y) + i\partial^m v_m(y)\right) - i(\overline{\theta}\overline{\theta})(\sigma^m\overline{\sigma}^n\theta)_\alpha \partial_m v_n(y) \\ &+ (\theta\theta)(\overline{\theta}\overline{\theta})\sigma^m_{\alpha\dot\beta}\partial_m\overline{\lambda}^{\dot\beta} \quad . \end{aligned} \tag{103}$$

A little more straightforward computation gives:

$$D_\alpha V_{WZ}^2 = \theta_\alpha(\overline{\theta}\overline{\theta})v^m v_m \quad , \tag{104}$$

as well as:

$$\begin{aligned} V_{WZ} D_\alpha V_{WZ} &= \frac{1}{2}\theta_\alpha(\overline{\theta}\overline{\theta})v^m v_m(y) + \frac{1}{4}\sigma^n_{\alpha\dot{\beta}}\bar{\sigma}^{m\dot{\beta}\gamma}\theta_\gamma(\overline{\theta}\overline{\theta})[v_m, v_n] \\ &\quad -\frac{1}{2}i(\theta\theta)(\overline{\theta}\overline{\theta})\sigma^m_{\alpha\dot{\beta}}[v_m, \bar{\lambda}^{\dot{\beta}}] \quad . \end{aligned} \tag{105}$$

Putting it all together, we have:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \sigma_\alpha^{mn\beta}\theta_\beta F_{mn}(y) + (\theta\theta)\sigma^m_{\alpha\dot{\beta}}\, \nabla_m \bar{\lambda}^{\dot{\beta}}(y) \quad , \tag{106}$$

where

$$\begin{aligned} F_{mn} &= \partial_m v_n - \partial_n v_m + i[v_m, v_n] \quad , \\ \nabla_m \bar{\lambda}^{\dot{\beta}} &= \partial_m \bar{\lambda}^{\dot{\beta}} + i[v_m, \bar{\lambda}^{\dot{\beta}}] \quad ; \end{aligned} \tag{107}$$

F_{mn} is the Yang-Mills field strength, while ∇_m is the Yang-Mills gauge covariant derivative.

We also need

$$W^\alpha = -\frac{1}{8}(\overline{D}\overline{D})e^{-2V} D^\alpha e^{2V} \quad ; \tag{108}$$

raising the index on Eq. 106 and Fierzing, we get:

$$W^\alpha = -i\lambda^\alpha(y) + \theta^\alpha D(y) + \theta^\beta\sigma_\beta^{mn\alpha} F_{mn}(y) - (\theta\theta)\bar{\sigma}^{m\dot{\beta}\alpha}\,\nabla_m \bar{\lambda}_{\dot{\beta}}(y) \quad . \tag{109}$$

3.5 N=1 linear multiplet

In the previous subsection we obtained the field strength multiplet by starting with the chiral spinor superfield W_α, and imposing the additional covariant constraint Eq. 85. Let us again start with a chiral spinor superfield Φ_α, $\overline{\Phi}_{\dot{\alpha}} = (\Phi_\alpha)^\dagger$, and construct a new superfield $L(x, \theta, \overline{\theta})$ as follows:

$$L(x, \theta, \overline{\theta}) = i\left(D^\alpha\Phi_\alpha + \overline{D}_{\dot{\alpha}}\overline{\Phi}^{\dot{\alpha}}\right) \quad . \tag{110}$$

The superfield $L(x, \theta, \overline{\theta})$ is real, since

$$\left(\overline{D}_{\dot{\alpha}}\overline{\Phi}^{\dot{\alpha}}\right)^\dagger = D_\alpha\Phi^\alpha = -D^\alpha\Phi_\alpha \quad ; \tag{111}$$

so $L(x, \theta, \overline{\theta})$ is a vector superfield which satisfies two additional covariant constraints:

$$(DD)L = (\overline{DD})L = 0 \quad . \tag{112}$$

These constraints follow trivially from the fact that Φ is chiral, and $(D)^3 = (\overline{D})^3 = 0$.

The component fields of $L(x, \theta, \overline{\theta})$ comprise the **linear multiplet**. These are a real scalar $C(x)$, a complex left-handed Weyl spinor χ_α, and a real divergenceless vector field A_m, $\partial^m A_m = 0$. Thus the linear multiplet has 1+4+3 = 8 real components.

4 N=1 Globally Supersymmetric Actions

Recall from the previous section that both the F component of a chiral superfield and the D component of a vector superfield transform by a total derivative under an N=1 supersymmetry transformation. Thus we immediately deduce two classes of N=1 globally supersymmetric actions:

$$\int d^4x \left[\int d^2\theta \, \Phi(y, \theta) + \int d^2\overline{\theta} \, \Phi^\dagger(y^\dagger, \overline{\theta}) \right] \tag{113}$$

is an invariant real action for **any** chiral superfield $\Phi(y, \theta)$, while

$$\int d^4x \int d^4\theta \, V(x, \theta, \overline{\theta}) \tag{114}$$

is an invariant real action for **any** vector superfield $V(x, \theta, \overline{\theta})$.

4.1 Chiral superfield actions

The Wess-Zumino model [14] is the simplest (sensible) N=1 SUSY model in four dimensions. The action is

$$\int d^4x \int d^4\theta \, \Phi^\dagger \Phi - \int d^4x \left[\int d^2\theta \, (\frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3) + \text{h.c.} \right] \quad , \tag{115}$$

where Φ is a chiral superfield.

Let us work out the part of this action containing bosonic component fields. The bosonic components of Φ and $\Phi^\dagger$ are:

$$\begin{aligned} \Phi(y, \theta) &= A(x) + \theta\theta F(x) + i\theta\sigma^m\overline{\theta}\partial_m A(x) \\ &\quad - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box A(x) \quad , \\ \Phi^\dagger(y^\dagger, \overline{\theta}) &= A^*(x) + \overline{\theta\theta}F^*(x) - i\theta\sigma^m\overline{\theta}\partial_m A^*(x) \\ &\quad - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box A^*(x) \quad . \end{aligned} \tag{116}$$

Thus:

$$\Phi^\dagger\Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} = -\frac{1}{4}\Box A^* A - \frac{1}{4}A^*\Box A + F^* F + \frac{1}{2}\partial^m A^* \partial_m A \quad , \tag{117}$$

where to obtain the last term we have used the Fierz identity Eq. 247.

We also have

$$\left[\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\right]_{\theta\theta} = mAF + gA^2F \quad , \tag{118}$$

so the part of the Wess-Zumino action containing only bosonic fields is:

$$\int d^4x \,\left[\partial^m A^* \partial_m A + F^* F - (mAF + gA^2F + \text{h.c.})\right] \tag{119}$$

We immediately notice that this action contains no derivatives acting on $F(x)$, i.e. $F(x)$ is an **auxiliary field** which can be eliminated by solving its equations of motion:

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta F} &= F^* - mA - gA^2 = 0 \quad , \\ \frac{\delta\mathcal{L}}{\delta F^*} &= F - mA^* - g(A^*)^2 = 0 \quad . \end{aligned} \tag{120}$$

This means we can write the bosonic part of the Wess-Zumino action as just

$$\int d^4x \,\left[\partial^m A^* \partial_m A - V(A, A^*)\right] \quad , \tag{121}$$

where the scalar potential $V(A, A^*)$ is given by:

$$V(A, A^*) = |F|^2 = [mA^* + g(A^*)^2][mA + gA^2] \quad . \tag{122}$$

More generally we could write

$$\int d^4x \int d^4\theta \,\Phi^\dagger\Phi - \int d^4x \left[\int d^2\theta \,W(\Phi) + \text{h.c.}\right] \quad , \tag{123}$$

where the **superpotential** $W(\Phi)$ is a **holomorphic function** of Φ, i.e. a functional only of Φ, not $\Phi^\dagger$. In this more general case the scalar potential is

$$V_F(A, A^*) = |F|^2 = \left|\frac{\delta W}{\delta\Phi}\right|^2_{\Phi=A} \quad . \tag{124}$$

Note that the scalar potential is obviously positive definite.

Since a cubic superpotential leads to a quartic scalar potential, we also see that the Wess-Zumino model is the most general unitary, **renormalizable** four-dimensional SUSY action for a single chiral superfield.

An even more general construction than Eq. 123 is

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \Phi^{j\dagger}) - \left[\int d^2\theta\, W(\Phi^i) + \text{h.c.}\right] \quad , \tag{125}$$

where $K(\Phi^i, \Phi^{j\dagger})$ is called the **Kähler potential**, and we now have an arbitrary number of chiral superfields Φ^i. The Kähler potential is a vector superfield: unlike the superpotential it is obviously not a holomorphic function of the Φ^i.

From the component expansion Eq. 71 it is clear that the Kähler potential produces kinetic terms with no more than two spacetime derivatives. If we replace some of the Φ^i by covariant derivatives of superfields, we will either obtain a higher derivative theory, or a theory which can be collapsed back to the form Eq.125. Thus if we exclude higher derivative theories Eq.125 is the most general action for (not necessarily renormalizable) N=1 SUSY models constructed from chiral superfields.

4.2 N=1 supersymmetric nonlinear sigma models

Bosonic nonlinear sigma models in D-dimensional spacetime have an action of the form:

$$\frac{1}{2}\int d^D x\, g_{ij}(A)\, \partial^m A^i(x)\partial_m A^j(x) \quad , \tag{126}$$

where the $A^i(x)$ are real scalar fields. The functional $g_{ij}(A)$ can be thought of as the **metric** of a **target space** Riemannian manifold with line element

$$ds^2 = g_{ij}\, dA^i dA^j \quad . \tag{127}$$

Nonlinear sigma models are not in general renormalizable, except in the case $D = 2$ with g_{ij} the metric of a symmetric space. [15]

The general chiral superfield action Eq.125 defines the supersymmetrized version of 4-dimensional nonlinear sigma models. [16] To see this, note that Eq. 117 implies that the kinetic term for the complex scalar components $A^i(x)$ is

$$g_{ij^*}\, \partial_m A^i(x)\partial^m A^{*j} \quad , \tag{128}$$

where:

$$g_{ij^*} \equiv \frac{\delta^2 K(A^i, A^{*j})}{\delta A^i \delta A^{*j}} \quad . \tag{129}$$

Since the Kähler potential is real. the target space metric g_{ij^*} is hermitian. To obtain a correct sign kinetic term for every nonauxiliary scalar field. we must also require that g_{ij^*} is positive definite and nonsingular: this implies (mild) restrictions on the choice of the Kähler potential.

A complex Riemannian manifold possessing a positive definite nonsingular hermitian metric which can be written (locally) as the second derivative of a scalar function is called a **Kähler manifold.** Thus Eq.125 defines supersymmetric generalized nonlinear sigma models whose target spaces are Kähler manifolds.

This is a rather powerful observation. since it implies that models with horrendously complicated component field Lagrangians can be characterized by the algebraic geometry of the target space. As an example. we will discuss the possible **holonomy groups** of sigma model target spaces.

Consider the parallel transport of a vector around a contractible closed loop using the Riemannian connection in a D-dimensional Riemannian space. The transported vector is related to the original vector by some $SO(D)$ rotation. The $SO(D)$ matrices obtained this way form a group, the local holonomy group of the manifold. Obviously the holonomy group is either $SO(D)$ itself or a subgroup of it. Four important examples are given below (we use the convention that $Sp(2D)$ is the symplectic group of rank D):

Manifold	**Maximum Holonomy Group**
General Riemannian space with real dimension D:	$SO(D)$
Kähler manifold with complex dimension D. real dimension $2D$:	$U(D)$
HyperKähler manifold with real dimension $4D$:	$Sp(2D)$
Quaternionic manifold with real dimension $4D$:	$Sp(2D) \times Sp(2)$

Note that the Kähler structure Eq. 129 (and thus also the action) is invariant under a **Kähler transformation:**

$$K(A^i, A^{*j}) \to K(A^i, A^{*j}) + \Lambda(A^i) + \Lambda^\dagger(A^{*j}) \quad . \tag{130}$$

It is also clear that both the Kähler structure Eq. 129 and the Riemannian structure Eq. 127 are preserved by arbitrary **holomorphic** transformations of the target space coordinates A^i.

4.3 N=1 supersymmetric Yang-Mills theory

We recall that W_α is a chiral spinor superfield and that a gauge transformation on the vector component of W_α is induced by the superfield transformation

$$W_\alpha \to e^{-i2\Lambda} W_\alpha e^{+i2\Lambda} \quad . \tag{131}$$

It follows that a gauge invariant supersymmetric action is

$$\begin{aligned} &\frac{1}{2}\int d^4x \int d^2\theta \,\mathrm{tr}\, W^\alpha W_\alpha \qquad (132)\\ &= \int d^4x \,\mathrm{tr}\left[-\frac{1}{4}F_{mn}F^{mn} - \frac{i}{4}F_{mn}\tilde{F}^{mn} - i\lambda\sigma^m \nabla_m \bar{\lambda} + \frac{1}{2}D^2\right] \quad , \end{aligned}$$

where we have used the explicit component expansions Eqs. 106.109. The dual field strength is defined as:

$$\tilde{F}^{mn} \equiv \frac{1}{2}\epsilon^{mnpq}F_{pq} \quad . \tag{133}$$

This action is not real and lacks any dependence upon the Yang-Mills gauge coupling g. The duality-friendly way to remedy these deficiencies is by introducing a complex gauge coupling τ:

$$\tau = \frac{\theta_{\mathrm{YM}}}{2\pi} + \frac{4\pi i}{g^2} \quad , \tag{134}$$

where θ_{YM} is the Yang-Mills theta parameter. The N=1 Yang-Mills action we want is then

$$\frac{1}{8\pi}\mathrm{Im}\left[\tau\int d^4x \int d^2\theta \,\mathrm{tr}\, W^\alpha W_\alpha\right] \quad . \tag{135}$$

$$\begin{aligned} &= \frac{1}{g^2}\int d^4x \,\mathrm{tr}\left[-\frac{1}{4}F_{mn}F^{mn} - i\lambda\sigma^m \nabla_m \bar{\lambda} + \frac{1}{2}D^2\right] \qquad (136)\\ &\qquad -\frac{\theta_{\mathrm{YM}}}{32\pi^2}\int d^4x \,\mathrm{tr}\, F_{mn}\tilde{F}^{mn} \quad . \end{aligned}$$

The minus sign in front of the θ_{YM} term is correct given the minus sign convention of Eq. 47.

Under a gauge transformation, chiral superfields Φ in the adjoint representation transform as:

$$\begin{aligned} \Phi &\to e^{-i2\Lambda}\Phi \quad ,\\ \Phi^\dagger &\to \Phi^\dagger e^{i2\Lambda^\dagger} \quad . \end{aligned} \tag{137}$$

Thus $\operatorname{tr}\Phi^\dagger\Phi$ is not gauge invariant. However from Eq. 97 we see that the following is a gauge invariant kinetic term for chiral superfields:

$$\operatorname{tr}\Phi^\dagger e^{2V}\Phi \quad . \tag{138}$$

In fact this is gauge invariant for Φ in an arbitrary representation R, not just the adjoint. In this case $\Lambda = t^a_{ij}\Lambda_a$, where the t^a_{ij} are matrices in the representation R. Thus Eq. 138 is still gauge invariant provided that all the tensor products contain the singlet. This is indeed true because the tensor product of R with its conjugate $\overline{R}$ contains both the singlet and adjoint representations, while every term in the series expansion of $\exp(2V)$ also contains either the singlet or the adjoint (or both).

Thus, supposing I have chiral superfields Φ^i transforming in representations R_i, the gauged version of the Wess-Zumino action is:

$$\begin{aligned} &\operatorname{tr}\int d^4x \int d^4\theta \, \Phi^{i\dagger} e^{2V}\Phi^i \\ &\qquad -\operatorname{tr}\int d^4x \left[\int d^2\theta \left(\frac{1}{2}m_{ij}\Phi^i\Phi^j + \frac{1}{3}g_{ijk}\Phi^i\Phi^j\Phi^k\right) + \text{h.c.}\right] \quad . \end{aligned} \tag{139}$$

Note that by gauge invariance m_{ij} can only be nonvanishing if

$$R_i = \overline{R}_j \quad . \tag{140}$$

Similarly, g_{ijk} can only be nonvanishing if $R_i \times R_j \times R_k$ contains the singlet.

The gauged kinetic term Eq. 138 contains a D-term $A^* DA$. The only other dependence on the auxiliary field D is the term $D^2/2g^2$ in the Yang-Mills action. Thus when we eliminate this auxiliary field by its equation of motion we find

$$\begin{aligned} D_a &= -g^2 A_b A_c^* \operatorname{tr}(T^a T^b T^c) \\ &= -\frac{1}{2} i g^2 f^{abc} A_b A_c^* \quad , \end{aligned} \tag{141}$$

where the second line follows from the fact that the adjoint representation is always anomaly-free. Thus:

$$D = T^a D_a = \frac{1}{2}g^2[A, A^*] \quad . \tag{142}$$

This implies that in the coupled Yang-Mills-Wess-Zumino model there is a new contribution to the scalar potential:

$$V_D = \frac{1}{2g^2}D^2 = \frac{g^2}{8}\left([A^i, A^{i*}]\right)^2 \quad . \tag{143}$$

So altogether the complete scalar potential is the sum of positive definite F and D-term contributions:

$$V(A^i, A^{i*}) = V_F + V_D = |F|^2 + \frac{1}{2g^2} D^2 \tag{144}$$

If we forget about renormalizablity we can write a very general N=1 action by gauging Eq. 125:

$$\begin{aligned}\mathcal{L} = &\int d^4\theta\, K(\Phi^i, \Phi^{j\dagger} e^{2V}) - \left[\int d^2\theta\, W(\Phi^i) + \text{h.c.}\right] \\ &+\frac{1}{8\pi}\text{Im}\left[\tau \int d^4x \int d^2\theta\, \text{tr}\, f(\Phi^i) W^\alpha W_\alpha\right] \quad ,\end{aligned} \tag{145}$$

where $f(\Phi^i)$ is a new holomorphic function called the **gauge kinetic function.** Note that every term in $f(\Phi^i)$ must transform like a representation which is contained in the tensor product of two adjoints.

5 N=2 Globally Supersymmetric Actions

5.1 N=2 superspace

There are several different ways to extend our treatment of N=1 rigid superspace to the case of N=2 rigid superspace. [17] Some methods, e.g. **harmonic superspace**, build in the $SU(2)$ automorphism symmetry of the N=2 generators Q^1_α, Q^2_α.

We will make do with the most naive extension of N=1 to N=2 superspace parameterizations:

$$\begin{aligned} \theta^\alpha,\, \overline{\theta}_{\dot\alpha} &\rightarrow \theta^\alpha,\, \overline{\theta}_{\dot\alpha},\, \tilde\theta^\alpha,\, \overline{\tilde\theta}_{\dot\alpha} \\ D_\alpha &\rightarrow D_\alpha,\, \tilde D_\alpha \\ \overline{D}_{\dot\alpha} &\rightarrow \overline{D}_{\dot\alpha},\, \overline{\tilde D}_{\dot\alpha} \\ \int d^2\theta &\rightarrow \int d^2\theta d^2\tilde\theta \end{aligned} \tag{146}$$

If we want to restore the $SU(2)$ global R symmetry, we should think of $(\theta^\alpha, \tilde\theta^\alpha)$ etc. as $SU(2)$ doublets.

5.2 N=2 chiral superfields

An N=2 chiral superfield $\Psi(x, \theta, \overline{\theta}, \tilde\theta, \overline{\tilde\theta})$ is defined as an N=2 scalar superfield which is a singlet under the global $SU(2)$ and which satisfies the covariant constraints

$$\begin{aligned} \overline{D}_{\dot{\alpha}}\Psi(x,\theta,\overline{\theta},\tilde{\theta},\overline{\tilde{\theta}}) &= 0 \quad , \\ \overline{\tilde{D}}_{\dot{\alpha}}\Psi(x,\theta,\overline{\theta},\tilde{\theta},\overline{\tilde{\theta}}) &= 0 \quad . \end{aligned} \tag{147}$$

It is convenient to introduce new bosonic coordinates

$$\tilde{y}^m = x^m + i\theta\sigma^m\overline{\theta} + i\tilde{\theta}\sigma^m\overline{\tilde{\theta}} \quad , \tag{148}$$

which obviously satisfy

$$\overline{D}_{\dot{\alpha}}\tilde{y}^m = \overline{\tilde{D}}_{\dot{\alpha}}\tilde{y}^m = 0 \quad . \tag{149}$$

If we expand an N=2 chiral superfield in powers of $\tilde{\theta}$, the components are N=1 chiral superfields. Thus:

$$\Psi = \Phi(\tilde{y},\theta) + i\sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y},\theta) + \tilde{\theta}\tilde{\theta}G(\tilde{y},\theta) \quad , \tag{150}$$

where $\Phi(y+i\tilde{\theta}\sigma^m\overline{\tilde{\theta}},\theta)$ and $G(y+i\tilde{\theta}\sigma^m\overline{\tilde{\theta}},\theta)$ are N=1 chiral superfields (note the effective "y" coordinate is shifted by $i\tilde{\theta}\sigma^m\overline{\tilde{\theta}}$), and $W_\alpha(y+i\tilde{\theta}\sigma^m\overline{\tilde{\theta}},\theta)$ is an N=1 chiral spinor superfield.

Since Ψ is an $SU(2)$ singlet, while $(\theta^\alpha,\tilde{\theta}^\alpha)$ is an $SU(2)$ doublet, it follows that the fermionic components ψ of Φ and λ of W_α also form an $SU(2)$ doublet. On the other hand the bosonic component fields A of Φ and v_m of W_α are $SU(2)$ singlets.

5.3 N=2 supersymmetric Yang-Mills theory

Suppose we write

$$\Psi(\tilde{y},\theta) \rightarrow T^a_{ij}\Psi_a(\tilde{y},\theta) \quad . \tag{151}$$

Then, since

$$\Psi^2\Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} = W^\alpha W_\alpha\Big|_{\theta\theta} + 2G\Phi\Big|_{\theta\theta} \tag{152}$$

the obvious form for N=2 Yang-Mills theory is:

$$\frac{1}{4\pi}\mathrm{Im}\left[\tau\int d^4x\int d^2\theta d^2\tilde{\theta}\ \mathrm{tr}\ \frac{1}{2}\Psi^2\right] \quad . \tag{153}$$

This clearly describes an N=1 Yang-Mills theory coupled to chiral superfields in the adjoint representation. Unfortunately something is wrong, since the second term in Eq. 152 is not a sensible Lagrangian for chiral superfields.

Clearly what we want is to be able to regard $G(\tilde{y},\theta)$ as an auxiliary superfield, and thus eliminate it in favor of Φ and V, reproducing (at least) the N=1 gauge invariant kinetic term Eq. 138.

Thus, while Eq. 153 is the correct action for N=2 Yang-Mills theory, we must impose additional covariant constraints on the N=2 chiral superfield Ψ. The correct constraints turn out to be:

$$(D^{a\alpha}D^b_\alpha)\Psi = (\overline{D}^a_{\dot\alpha}\overline{D}^{b\dot\alpha})\Psi^\dagger \quad , \tag{154}$$

where a, b are global $SU(2)$ indices:

$$\begin{aligned} D^a &= (D, \tilde{D}) \quad , \\ \overline{D}^a &= (\overline{D}, \overline{\tilde{D}}) \quad . \end{aligned} \tag{155}$$

Rather than solve these constraints directly, it is much easier to simply assume that G can be eliminated in favor of Φ and V, then deduce the correct expression from the requirement of gauge invariance (i.e. gauge invariance in the N=1 sense). Roughly speaking, we need something like

$$G(\tilde{y},\theta) \sim \Phi^\dagger e^{2V} \quad . \tag{156}$$

However, while the right-hand side transforms correctly under gauge transformations, it is clearly **not** an N=1 chiral superfield. So consider instead the more sophisticated expression:

$$G(\tilde{y},\theta) = \int d^2\overline{\theta}\ \Phi^\dagger(\tilde{y} - i\theta\sigma\overline{\theta}, \overline{\theta}) e^{2V(\tilde{y}-i\theta\sigma\overline{\theta},\theta,\overline{\theta})} \quad , \tag{157}$$

where the integral is meant to be performed for **fixed** $\tilde{y}$.

The result of the integral is obviously a function only of $\tilde{y}$ and θ, so $G(\tilde{y},\theta)$ thus defined is an N=1 chiral superfield, as required. Under the N=1 superfield transformation which induces a gauge transformation, the integrand of Eq. 157 transforms as:

$$\begin{aligned} \Phi^\dagger e^{2V} &\rightarrow \Phi^\dagger e^{2V} e^{i2\Lambda(\tilde{y}-i\theta\sigma\overline{\theta}+i\theta\sigma\overline{\theta},\theta)} \\ &= \Phi^\dagger e^{2V} e^{i2\Lambda(\tilde{y},\theta)} \quad , \end{aligned} \tag{158}$$

so we can pull the $\exp(i2\Lambda(\tilde{y},\theta)$ factor out of the integral. Thus

$$G(\tilde{y},\theta) \rightarrow G(\tilde{y},\theta) e^{i2\Lambda(\tilde{y},\theta)} \quad , \tag{159}$$

as required for gauge invariance.

The overall coefficient of 1 in Eq. 157 is fixed by the global $SU(2)$ symmetry. As we noted above the fermionic components ψ of Φ and λ of W_α form an $SU(2)$ doublet. Thus the relative coefficient of the kinetic terms for ψ in $G\Phi$ and λ in $W^\alpha W_\alpha$ must be equal.

The resulting N=2 Yang-Mills theory is thus equivalent to N=1 Yang-Mills coupled to matter fields in the adjoint representation. There is no superpotential, but there is a scalar potential coming from the D-term. The nonauxiliary fields form an off-shell N=2 vector multiplet: v_m, A, and the global $SU(2)$ doublet (ψ, λ). On-shell this multiplet gives 4+4 = 8 real field components, which of course agrees with the counting for the massless N=2 vector multiplet of single particle states.

5.4 The N=2 prepotential

If we forget about renormalizability, we can write a much more general action for N=2 chiral superfields satisfying the covariant constraint Eq. 154:

$$\frac{1}{4\pi}\mathrm{Im}\left[\int d^4x \int d^2\theta d^2\tilde{\theta}\ \mathrm{tr}\,\mathcal{F}(\Psi)\right] \quad , \tag{160}$$

where the holomorphic functional $\mathcal{F}(\Psi)$ is called **the N=2 prepotential.** Obviously

$$\mathcal{F}(\Psi) = \frac{1}{2}\tau\Psi^2 \tag{161}$$

gives back the classical N=2 Yang-Mills action of Eq. 153.

Let us define

$$\begin{aligned}\mathcal{F}_a(\Phi) &= \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi_a} \quad , \\ \mathcal{F}_{ab}(\Phi) &= \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi_a\partial\Phi_b} \quad .\end{aligned} \tag{162}$$

Then the general Lagrangian can be written in terms of N=1 superfields as follows:

$$\frac{1}{4\pi}\mathrm{Im}\left[\int d^2\theta\ \frac{1}{2}\mathcal{F}_{ab}(\Phi)W^{\alpha a}W^b_\alpha + \int d^4\theta\ (\Phi^\dagger e^{2V})^a\mathcal{F}_a(\Phi)\right] \quad . \tag{163}$$

Thus from the N=1 point of view we have a special case of Eq. 145: the superpotential vanishes, the Kähler potential is

$$K = \mathrm{Im}\left[(\Phi^\dagger e^{2V})^a\mathcal{F}_a(\Phi)\right] \quad , \tag{164}$$

and the gauge kinetic function is

$$f(\Phi) = \mathcal{F}_{ab} T^a T^b \quad . \tag{165}$$

Notice that in this more general N=2 action the scalar fields A^a describe a nonlinear sigma model whose target space Kähler potential has the special form above, i.e. it can be written in terms of a derivative of a holomorphic function. The target space is a special Kähler manifold known as the "special Kähler" manifold. [18]

5.5 N=2 hypermultiplets

While Ψ was assumed to be a singlet under the global $SU(2)$ symmetry, we can also consider a general N=2 scalar superfield which is an $SU(2)$ doublet:

$$\Phi^a(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}}) \quad .$$

An N=2 hypermultiplet superfield is then defined by the covariant constraints

$$\begin{aligned} D^a_\alpha \Phi^b &= \frac{1}{2}\epsilon^{ab} D^c_\alpha \Phi^c \quad , \\ \overline{D}^a_{\dot\alpha} \Phi^b &= \frac{1}{2}\epsilon^{ab} \overline{D}^c_{\dot\alpha} \Phi^c \quad . \end{aligned} \tag{166}$$

These constraints simply remove the isotriplet parts of $D^a_\alpha \Phi^b$ and $\overline{D}^a_{\dot\alpha} \Phi^b$:

$$[\frac{1}{2}] + [\frac{1}{2}] = [0]_{\mathbf{antisymm.}} + [1]_{\mathbf{symm.}} \quad . \tag{167}$$

The independent component fields are:

$A^a(x)$,	complex scalar isodoublet
$\psi_\alpha(x)$, $\chi_\alpha(x)$,	two isosinglet spinors
$F^a(x)$,	complex auxiliary scalar isodoublet

On-shell this implies 4+4 = 8 real components, which as we have already noted is twice the number in the massless N=2 hypermultiplet of single particle states.

A free superspace action for an N=2 hypermultiplet superfield Φ^a is

$$\int d^4x \, D^{\alpha a} D^b_\alpha \left[\Phi^{a\dagger} D^{\beta c} D^c_\beta \Phi^b \right] \quad . \tag{168}$$

With more difficulty, we can couple N=2 hypermultiplets to N=2 Yang-Mills: the details are not particularly illuminating.

Note that there can be no renormalizable self-interaction for Φ^a since there is no cubic $SU(2)$ invariant.

We can construct N=2 generalizations of nonlinear sigma models out of the hypermultiplets. It is easiest to start with the N=1 case in components

$$g_{ij^*}\partial_m A^i(x)\partial^m A^{*j} + \dots \quad , \tag{169}$$

then impose the extra constraints of N=2 supersymmetry. The end result [19] is that the target spaces of N=2 hypermultiplet nonlinear sigma models are **hyperKähler** manifolds.

6 Supergravity

So far we have only considered global supersymmetry, generated by

$$\xi Q + \overline{\xi Q}$$

with ξ^α, $\bar{\xi}_{\dot\alpha}$ **constant** Grassmann parameters. If we want **local supersymmetry**, we should promote these parameters to functions of spacetime:

$$\xi^\alpha,\ \bar{\xi}_{\dot\alpha} \rightarrow \xi^\alpha(x),\ \bar{\xi}_{\dot\alpha}(x) \quad . \tag{170}$$

Rigid superspace then becomes **curved superspace.** From the superspace vielbein E_M^A and spin connection W_A^{mn} we can construct the superspace curvature and torsion:

$$R_{MNA}{}^B, \quad T_{MN}{}^A \quad .$$

Recall that N=1 rigid superspace has already nonzero torsion, so we **cannot** constrain all components of the curved superspace torsion to vanish as we do in general relativity. On the other hand the superspace vielbein and connection have too many independent components to define a sensible theory. Thus the main difficulty in constructing supergravity theories is finding and solving an appropriate set of covariant constraints. This gets very complicated.[20] and is beyond the scope of these lectures.

Let us instead quote results. One can construct an off-shell supergravity multiplet with the following field content:

e_m^a ,	vierbein $\rightarrow$ spin 2 graviton
ψ_m^α ,	vector-spinor $\rightarrow$ spin 3/2 gravitino
b_a ,	auxiliary real vector field
M ,	auxiliary complex scalar field

From these fields we can construct a supergravity Lagrangian:

$$
\begin{aligned}
8\pi G\mathcal{L} = & -\frac{1}{2}eR - \frac{1}{3}e|M|^2 + \frac{1}{3}eb^a b_a \\
& +\frac{1}{2}e\epsilon^{klmn}\left(\bar{\psi}_k\bar{\sigma}_l\mathcal{D}_m\psi_n - \psi_k\sigma_l\mathcal{D}_m\bar{\psi}_n\right)
\end{aligned}
\tag{171}
$$

where:

$$
\begin{aligned}
G &= \text{Newtons constant,} \\
e &= \det e_m^a, \\
R &= \text{Ricci scalar curvature,} \\
\mathcal{D}_m &= \text{covariant derivative for spin 3/2 fields.}
\end{aligned}
$$

The action is invariant under

- general coordinate transformations,
- local Lorentz transformations,
- local N=1 supersymmetry.

Let's count the off-shell degrees of freedom of N=1 supergravity. Because the action is invariant under three types of local symmetries we should only count gauge invariant degrees of freedom:

$$
\begin{aligned}
e_m^a :\quad & 4 \times 4 = 16 \\
& -4 \text{ general coordinate } \zeta^m \\
& -6 \text{ local Lorentz } \lambda^{ab} \\
& = 6 \text{ bosonic real components}
\end{aligned}
$$

$$
\begin{aligned}
\psi_m^\alpha :\quad & 4 \times 4 = 16 \\
& -4 \text{ local } N = 1 \text{ SUSY } \xi^\alpha \\
& = 12 \text{ fermionic real components}
\end{aligned}
$$

$$
\begin{aligned}
b_a :\quad & \text{real vector} \\
& = 4 \text{ bosonic real components}
\end{aligned}
$$

$$
\begin{aligned}
M :\quad & \text{complex scalar} \\
& = 2 \text{ bosonic real components}
\end{aligned}
$$

Thus we have a total of 12 bosonic and 12 fermionic real components in the off-shell N=1 supergravity multiplet. On-shell we have only 2+2 components, corresponding to a massless spin 2 graviton and a massless spin 3/2 gravitino.

Of course, we really want to be able to couple N=1 supergravity to N=1 supersymmetric Yang-Mills and N=1 chiral superfield matter, all in a way which is consistent with local supersymmetry. This is again a complicated problem and the final result is not particularly intuitive. [2]

We can also extend 4-dimensional N=1 supergravity to N=2, 3, 4, or 8 supergravity. These extended supergravities automatically couple gravity to gauge fields and matter fields in a way consistent with local supersymmetry, just as N=2 Yang-Mills couples gauge fields to matter in a way consistent with global supersymmetry. Extended supergravities are easier to construct and understand if we use dimensional reduction. For example, 4-dimensional N=8 supergravity can be obtained by dimensionally reducing 11-dimensional N=1 supergravity, which is a rather simple theory to describe. We will return to this fact when we discuss supersymmetry in higher dimensions.

7 Renormalization of N=1 SUSY Theories

Consider again the Wess-Zumino model:

$$\int d^4x \int d^4\theta\, \Phi^\dagger \Phi - \int d^4x \left[\int d^2\theta \left(\frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3 \right) + \text{h.c.} \right] \quad , \tag{172}$$

We would like to work out the superfield Feynman rules of this theory. However we encounter an immediate difficulty which is that Φ is not a general scalar superfield, but rather a constrained superfield. Thus in computing perturbative diagrams with chiral superfields we must deal with the occurence of integrals $\int d^4x \int d^2\theta$ over only part of the full N=1 rigid superspace.

This difficulty is overcome by introducing a **projection operator** for chiral superfields. The projection operator we need is

$$P_+ \equiv -\frac{1}{16\Box} \overline{D}^2 D^2 \quad . \tag{173}$$

This operator clearly has the property that it takes a general scalar superfield to a chiral superfield, i.e.

$$\overline{D}_{\dot{\alpha}} P_+ \Phi(x, \theta, \overline{\theta}) = 0 \tag{174}$$

follows trivially from the fact that $(\overline{D})^3 = 0$. To prove that P_+ is in fact a projection operator, we must also show that

$$(P_+)^2 = P_+ \quad .$$

We use the following identity (which can be verified by brute force):

$$[\overline{D}^2, D^2] = 8i(D\sigma^m\overline{D})\partial_m - 16\Box \quad . \tag{175}$$

Thus:

$$\begin{aligned}
(P_+)^2 &= \frac{1}{16\Box}\overline{D}^2 D^2 \frac{1}{16\Box}\overline{D}^2 D^2 \\
&= \left(\frac{1}{16\Box}\right)^2 \overline{D}^2 D^2 \overline{D}^2 D^2 \\
&= \left(\frac{1}{16\Box}\right)^2 \overline{D}^2 D^2 (D^2\overline{D}^2 + 8i(D\sigma^m\overline{D})\partial_m - 16\Box) \qquad (176) \\
&= \left(\frac{1}{16\Box}\right)^2 \overline{D}^2 D^2 (-16\Box) \\
&= P_+ \quad , \qquad (177)
\end{aligned}$$

where in the fourth line we used $(D)^4 = (D)^3 = 0$.

Similarly the projection operator for antichiral superfields is

$$P_- \equiv -\frac{1}{16\Box} D^2\overline{D}^2 \quad . \tag{178}$$

We can now deal with the occurence of integrals $\int d^4x \int d^2\theta$ over only part of the full N=1 rigid superspace. The judicious use of projection operator insertions allow us to convert these into integrals over the entire superspace. For example:

$$\begin{aligned}
\int d^2\theta \; m\Phi\Phi &= \int d^2\theta \; m\Phi P_+\Phi \\
&= -4\int d^4\theta \; m\Phi \frac{1}{16\Box} D^2\Phi \quad , \qquad (179)
\end{aligned}$$

where in the last line we used the fact that $\overline{D}^2\Phi = 0$ and that, modulo surface terms,

$$\int d^4x\,\overline{D}^2 \equiv -4\int d^4x \int d^2\overline{\theta} \quad . \tag{180}$$

A similar difficulty occurs for the cubic interaction term $g\Phi^3$. These vertices correspond to functional derivatives with respect to chiral sources $J(y,\theta)$. These functional derivatives produce superspace delta functions

$$\delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2)$$

whereas what we want (for internal lines anyway) are delta functions for the full superspace:

$$\begin{aligned}\delta^4(x_1 - x_2)\delta^4(\theta_1 - \theta_2) &= \delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2) \\ &= \delta^4(x_1 - x_2)(\theta_1 - \theta_2)^2(\bar{\theta}_1 - \bar{\theta}_2)^2 \qquad (181)\end{aligned}$$

This is easily remedied by using the identity

$$\delta^4(x_1 - x_2)\delta^2(\theta_1 - \theta_2) = -\frac{1}{4}\overline{D}^2\delta^4(x_1 - x_2)\delta^4(\theta_1 - \theta_2) \quad . \qquad (182)$$

In loop graphs, one factor of $\overline{D}^2$ from each vertex will get used up converting an $\int d^2\theta$ to an $\int d^4\theta$. Of course we also have a similar trick for the $g\Phi^{\dagger 3}$ vertices.

We can now compute the superspace propagator of the Wess-Zumino model by performing the functional integral of the quadratic part of the action, written in the form:

$$\int d^4x \int d^4\theta \left[\frac{1}{2}(\Phi\ \Phi^\dagger)\begin{pmatrix}\frac{1}{4}\frac{m}{\Box}D^2 & 1 \\ 1 & \frac{1}{4}\frac{m}{\Box}\overline{D}^2\end{pmatrix}\begin{pmatrix}\Phi \\ \Phi^\dagger\end{pmatrix} + (\Phi\ \Phi^\dagger)\begin{pmatrix}-\frac{1}{4}\frac{D^2}{\Box}J \\ -\frac{1}{4}\frac{\overline{D}^2}{\Box}\bar{J}\end{pmatrix}\right] \quad . \qquad (183)$$

7.1 *Nonrenormalization*

Without further ado we can now summarize **the superspace dependence** of the resulting Feynman rules for 1PI diagrams:

- There is an $\int d^4\theta$ for each vertex.
- For a Φ^3 vertex n of whose lines are external, there are $2-n$ factors of $\overline{D}^2$. For a $\Phi^{\dagger 3}$ vertex, there are $2-n$ factors of D^2.
- There is a Grassmann delta function

  $$\delta^4(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2(\bar{\theta}_1 - \bar{\theta}_2)^2 \qquad (184)$$

  for each propagator.
- There is a factor of D^2 for each Φ-Φ propagator, and a factor of $\overline{D}^2$ for each $\Phi^\dagger$-$\Phi^\dagger$ propagator.

Consider now an arbitrary loop graph. Integrating the various D^2 and $\overline{D}^2$ factors by parts, we can perform all but one of the $\int d^4\theta$ integrations using the delta functions. Let $\int d^4\theta$ denote the final integration, and $\delta^4(\theta - \theta_2)$ be the one remaining Grassmann delta function. This delta function is already supposed to be evaluated at $\theta = \theta_2$, due to the θ_2 integral already performed. So the graph vanishes unless there is precisely one factor of D^2 and one factor of $\overline{D}^2$ acting on the final delta function:

$$D^2\overline{D}^2\delta^4(\theta - \theta_2) = 16 \quad .$$

The only remaining θ dependence comes from the external lines. Thus an arbitrary term in the effective action can be reduced to the form:

$$\int d^4\theta \int d^4x_1 \cdots d^4x_n \; F_1(x_1, \theta, \overline{\theta}) \cdots F_n(x_n, \theta, \overline{\theta}) G(x_1, \ldots x_n) \quad , \qquad (185)$$

where the F's are superfields and covariant derivatives of superfields, and all the spacetime structure is swept into the translationally invariant function $G(x_1, \ldots x_n)$.

This result is called the **N=1 Nonrenormalization Theorem**. It generalizes to N=1 actions containing arbitrary numbers of chiral and vector superfields. An important consequence is that if all of the external lines are chiral, or if all of the external lines are antichiral, the expression above vanishes. Thus:

The superpotential is not renormalized at any order in perturbation theory.

Another important result of the above analysis is that all vacuum diagrams and tadpoles vanish. This is consistent with the fact that the vacuum energy is precisely zero in any globally supersymmetric theory.

Note our derivation implicitly assumed that the spacetime loop integrals are regulated in a way which is consistent with supersymmetry. This is **not** the case if we employ dimensional regularization, since the numbers of fermions and bosons vary differently as you vary the dimension. Supersymmetric loop diagrams are usually evaluated using a regularization called **dimensional reduction**, where the spinor algebra is fixed at four dimensions while momentum integrals are performed in $4 - 2\epsilon$ dimensions. This is not a completely satisfactory procedure either. [3]

7.2 *Renormalization*

What renormalization do we have to perform in an N=1 SUSY model with chiral superfields Φ^i and a vector superfield V? We have wave function renormalization:

$$\begin{aligned} \Phi_0^i &= Z^{1/2^{ij}}\,\Phi^j \quad , \\ V_0 &= Z_V^{1/2}\,V \quad , \end{aligned} \tag{186}$$

and we also have gauge coupling renormalization:

$$g_0 = Z_g\, g \quad . \tag{187}$$

Even better, if we compute using the background field method, the background field gauge invariance implies the relation: [3]

$$Z_g\, Z_V^{1/2} = 1 \quad . \tag{188}$$

The end result is that we can characterize the renormalized theory in terms of two objects:

- The beta function:
$$\beta(g) = \mu\frac{\partial}{\partial\mu}g \quad ;$$

- The anomalous dimensions matrix of the Φ^i:
$$\gamma^{ij} = Z^{-1/2^{ik}}\mu\frac{\partial}{\partial\mu}Z^{1/2^{kj}} \quad .$$

8 Holomorphy and the N=2 Yang-Mills Beta Function

In this section we will review some of Seiberg's original arguments about the N=2 supersymmetric $SU(2)$ Yang-Mills beta function. [21] This type of argumentation deals with the effective infrared (i.e. low energy) limit of the theory, described by the **Wilsonian action.** [22] The form of this effective action will be constrained by three kinds of considerations:

- holomorphy,
- global symmetries,
- the existence of a nonsingular weak coupling limit.

Let us begin by listing the global symmetries of the classical N=2 supersymmetric $SU(2)$ Yang-Mills Lagrangian. These are: the global $SU(2)$ R symmetry arising from the automorphism of the N=2 algebra, and an additional $U(1)$ R symmetry defined by:

$$\begin{aligned} \theta &\rightarrow e^{i\omega}\theta \quad , \\ \bar{\theta} &\rightarrow e^{i\omega}\bar{\theta} \quad , \qquad (189) \\ \Psi &\rightarrow e^{2i\omega}\Psi \quad . \qquad (190) \end{aligned}$$

There is an axial current j^R_m corresponding to the $U(1)$ R symmetry. Since both fermions ψ and λ have R-charge +1, there is a nonvanishing ABJ triangle anomaly. We write the anomalous divergence of the R current, remembering that the fermions are in the adjoint representation:

$$\partial^m j^R_m = 2C_2(G)\frac{g^2}{16\pi^2}F_{mn}\tilde{F}^{mn} = \frac{g^2}{4\pi^2}F_{mn}\tilde{F}^{mn} \quad . \qquad (191)$$

Next we deduce the **moduli space** of gauge inequivalent classical vacua for N=2 supersymmetric $SU(2)$ Yang-Mills. The theory contains an $SU(2)$ triplet complex scalar field $A(x)$ whose scalar potential is (see Eq. 143):

$$V(A) = \frac{1}{2g^2}D^2 = \frac{g^2}{8}\left([A, A^*]\right)^2 \quad . \qquad (192)$$

Unbroken supersymmetry requires that $V(A) = 0$ in the vacuum. Up to a gauge transformation, the most general solution to this requirement is:

$$A(x) = \frac{1}{2}a\sigma^3 \quad , \qquad (193)$$

where σ^3 is the Pauli matrix and a is a complex constant. The parameter a does not quite give only gauge inequivalent vacua, since by Weyl symmetry vacua labeled a and $-a$ are gauge equivalent. So the classical moduli space is described by a complex parameter u, with

$$u = \frac{1}{2}a^2 = \langle \mathrm{tr}\, A^2 \rangle \quad . \qquad (194)$$

For a generic nonvanishing value of u, the $SU(2)$ gauge symmetry is broken down to a $U(1)$. Since N=2 SUSY is still in force, masses are generated not just for two components of the $SU(2)$ gauge field, but also for their N=2 superpartners. Thus the remaining light fields consist of a $U(1)$ gauge boson, a massless uncharged complex scalar, and two massless uncharged fermions.

The infrared effective action clearly exists in this case. The point $u = 0$, on the other hand, appears singular.

The form of the infrared effective action is severely constrained by N=2 supersymmetry. The part of this action which contains no more than two spacetime derivatives and interactions of no more than four fermions must have the same form as the classical action of the ultraviolet theory:

$$\frac{1}{4\pi}\mathrm{Im}\left[\int d^4x \int d^2\theta d^2\tilde{\theta}\ \mathrm{tr}\,\mathcal{F}_{\mathrm{eff}}(\Psi)\right] \quad , \tag{195}$$

Because of the anomaly, the effective action is not invariant under a $U(1)_R$ transformation. Instead, the Adler-Bardeen theorem tells us that the effective action is shifted by

$$\begin{aligned} & \omega \int d^4x\ \frac{1}{4\pi^2} F_{mn}\tilde{F}^{mn} \\ = \quad & -\omega\ \mathrm{Im}\left[\int d^4x \int d^2\theta d^2\tilde{\theta}\ \frac{1}{2\pi^2}\Psi^2\right] \quad . \end{aligned} \tag{196}$$

Neglecting (for the moment) instanton effects, the effective action is still constrained by $U(1)_R$ invariance modulo this shift, which is manifestly a one-loop effect. The only holomorphic functional of Ψ with this property is

$$\mathcal{F}_1 = i\frac{1}{2\pi}\Psi^2 \ln\frac{\Psi^2}{\Lambda^2} \quad , \tag{197}$$

where Λ is a dynamically generated scale. Actually, since we can absorb the tree-level $\mathcal{F}$ into the definition of Λ, $\mathcal{F}_1$ is the full effective prepotential to all orders in perturbation theory.

From the shift Eq. 196 we see that a single instanton violates R charge conservation by 8 units, breaking the global $U(1)_R$ symmetry down to Z_8 (in fact, since u carries R charge 4, there is a further breaking down to Z_4). This suggests that the complete nonperturbative effective prepotential has the form:

$$\mathcal{F} = i\frac{1}{2\pi}\Psi^2 \ln\frac{\Psi^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{\Psi}\right)^{4k}\Psi^2 \quad , \tag{198}$$

where the $\mathcal{F}_k$ are numerical coefficients, and the kth term arises as a contribution of k instantons.

Returning to the all orders perturbative result, we can deduce the effective Wilsonian gauge coupling $g_W(u)$ from the gauge kinetic function:

$$f(\Phi) = \frac{\delta^2\mathcal{F}(\Phi)}{\delta\Phi\delta\Phi} \quad ; \tag{199}$$

thus:

$$\frac{1}{g_W(u)} = \frac{1}{g_0^2}\left[1 + \frac{3g_0^2}{4\pi^2} + \frac{g_0^2}{4\pi}\ln\left(\frac{u}{\Lambda^2}\right)\right] \quad . \tag{200}$$

This gives us the all-orders perturbative beta function:

$$\beta(g) = -\frac{g^3}{4\pi^2} \quad . \tag{201}$$

We could try to make an analogous derivation in the case of N=1 SUSY Yang-Mills. However in the N=1 case there is a separate wave function renormalization of the D-term of the chiral superfield action. Because of this $g_W \neq g_{\text{eff}}$ and Eq. 201 is not valid.

From the expression for the effective gauge coupling we see that the $u \to \infty$ limit is the weak coupling limit. This explains why we did not include negative k contributions to Eq. 198. i.e. why the sum extends only over positive k. Terms with negative k blow up like a power as $u \to \infty$, behavior which is inconsistent with the existence of a nonsingular weak coupling limit.

9 Supersymmetry in spacetime dimensions 2, 6, 10, and 11

9.1 *Spinors in arbitrary spacetime dimensions*

The dimension of Dirac spinors in d spacetime dimensions can be deduced by constructing the Dirac gamma matrices obeying the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \quad . \tag{202}$$

The result is

$$d_\gamma = \begin{cases} 2^{d/2} & \text{d even.} \\ 2^{(d-1)/2} & \text{d odd.} \end{cases} \tag{203}$$

Starting with Dirac spinors. we can investigate whether it is possible to impose Weyl, Majorana. or simultaneously Weyl and Majorana conditions on these spinors.

For Weyl spinors. we need to generalize the notion of the chirality operator γ^5. Recall that in four dimensions. CPT conjugate spinors have opposite chirality. implying that there are no gravitational anomalies. [23] This is related to the fact that γ^5. defined as

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \tag{204}$$

has eigenvalues $\pm i$. since

$$(\gamma^5)^2 = -I \quad . \tag{205}$$

Table 2: Properties of spinors in spacetime dimensions 2 to 12. d_γ is the dimension of Dirac gamma matrices.

d	2	3	4	5	6	7	8	9	10	11	12
d_γ	2	2	4	4	8	8	16	16	32	32	64
minimum spinor dim.	1	2	4	8	8	16	16	16	16	32	64
Weyl?	X		X		X		X		X		X
Majorana?	X	X	X				X	X	X	X	X
Majorana-Weyl?	X								X		
gravitational anomalies?	X				X				X		

For any spacetime dimension $d = 4k$, $k = 1, 2, \ldots$ we can define "γ^5" in an exactly analogous way:

$$\gamma^5 = \gamma^0\gamma^1 \cdots \gamma^{4k-1} \quad , \tag{206}$$

and Eq. 205 still holds. Thus in $d = 4k$ dimensions Weyl spinors exist and gravitational anomalies are absent. In $d = 4k+2$ dimensions

$$(\gamma^5)^2 = I \quad , \tag{207}$$

implying that CPT conjugate spinors have the same chirality. Thus Weyl spinors exist and gravitational anomalies are possible. In odd dimensions γ^5 defined as above is the identity; there is no chirality operator and thus no Weyl spinors.

The analysis of Majorana and Majorana-Weyl conditions in arbitrary dimensions is more involved: a good reference is Sohnius. [4] The results are summarized in Table 2.

9.2 Supersymmetry in arbitrary spacetime dimensions

To discuss supersymmetry in spacetime dimensions other than four, we need an improved notation for keeping track of the number of independent supersymmetry generators. In four dimensions N=1 SUSY means that there are four independent SUSY generators:

$$Q_1,\ Q_2,\ \overline{Q}_{\dot{1}},\ \overline{Q}_{\dot{2}} \quad .$$

This is of course the minimum number of supersymmetries in four dimensions since the minimum spinor dimension is four. Let us refer to this as $N=(1)_4$ supersymmetry. More generally, $N=(p)_{p*n_{\min}}$ denotes $p*n_{\min}$ supersymmetries, where $n_{\min}$ is the minimum spinor dimension.

In this new notation the possible global supersymmetries in four dimensions are:

$$\begin{aligned} N &= (1)_4 \\ N &= (2)_8 \\ N &= (4)_{16} \\ N &= (8)_{32} \\ &\ldots \end{aligned}$$

In $4k+2$ dimensions we can have **independent** chiral and antichiral SUSY generators, since CPT conjugates have the same chirality. Thus we need a notation which distinguishes chiral from antichiral SUSY generators:

$$N = (p,q)_{(p+q)*n_{\min}}$$

where p, q are the number of chiral/antichiral SUSY generators, respectively.

From Table 2 it is clear that in any spacetime dimension there is a **minimum** number of supersymmetries (other than zero). Thus:

As few as 4 supersymmetries can only occur for: . . . . $d \leq 4$

As few as 8 supersymmetries can only occur for: . . . . $d \leq 6$

As few as 16 supersymmetries can only occur for: . . . . $d \leq 10$

As few as 32 supersymmetries can only occur for: . . . . $d \leq 11$

Furthermore, in any spacetime dimension, the **maximum** number of supersymmetries of physical interest is always 32 or less. This is because for more than 32 supersymmetries all massless multiplets contain unphysical higher spin particles, i.e. particles with spin greater than that of the d-dimensional graviton.

Clearly we expect that many SUSY theories in different spacetime dimensions but with the same number of supersymmetries can be related, presumably through some form of dimensional reduction or truncation. Indeed this is true as we will see in several examples. Of particular interest is the possibility of relating models with extended supersymmetry in four dimensions to simpler models in the "mother" dimensions 6, 10, and 11.

9.3 Supersymmetry in 2 dimensions

In two dimensions we have (p,q) type superalgebras. I will briefly describe some examples.

- $(1,0)_1$ supersymmetry: here we have a single left-handed Majorana-Weyl spinor:

$$\begin{aligned} Q_+ &= Q_+^\dagger \quad , \\ \{Q_+, Q_+\} &= 2iP_z \quad , \qquad (208) \\ [Q_+, P_z] &= [Q_+, P_{\bar{z}}] = 0 \quad , \qquad (209) \end{aligned}$$

  where the antihermitian generators P_z, $P_{\bar{z}}$ generate left and right-moving translations in the two-dimensional spacetime parameterized by coordinates z, $\bar{z} = x^0 \pm x^1$. The minimal SUSY multiplet has just two states: one left-moving real scalar (a "chiral boson"), and one left-moving Majorana-Weyl fermion.

- $(1,1)_2$ supersymmetry: here we can construct a $(1,1)$ supersymmetric nonlinear sigma model by supersymmetrizing

$$\int dz d\bar{z}\, g_{ij}(A)\partial_z A^i \partial_{\bar{z}} A^j \quad . \qquad (210)$$

  The target spaces of such models are general Riemannian manifolds.

- $(2,2)_4$ supersymmetry: here again we can construct a supersymmetric nonlinear sigma model. The target spaces are Kähler manifolds. Note that this agrees with the four-dimensional N=$(1)_4$ case, which has the same number of supersymmetries.

- $(4,4)_8$ supersymmetry: here we have supersymmetric nonlinear sigma models whose target spaces are hyperKähler manifolds. Again this agrees with the four-dimensional N=$(2)_8$ case, which has the same number of supersymmetries.

Two-dimensional supersymmetry has important applications to superstring theory, where it is interpreted as **worldsheet supersymmetry** rather than spacetime SUSY. For more details see Hirosi Ooguri's lectures in this volume.

9.4 Supersymmetry in 6 dimensions

In six dimensions the minimal case is N=$(1,0)_8$ or $(0,1)_8$. The SUSY generators can be expressed as a single complex Weyl spinor:

$$\begin{aligned} Q_a \quad &, \quad a = 1,\ldots 8; \\ \{Q_a, \overline{Q}^b\} \quad &= \quad \frac{1}{2}(1+\gamma^7)_a^c(\gamma^M)_c^b P_M \quad , \end{aligned} \tag{211}$$

where we use capital Roman letters to denote 6-dimensional spacetime indices, and γ^7 is the chirality operator, i.e. the 6-dimensional version of "γ^5".

In six dimensions massless particles are labelled by irreps of the little group $Spin(4) \sim SU(2)\times SU(2)$. Let us describe the possible massless irreps of $(0,1)_8$ supersymmetry in terms of their $SU(2)\times SU(2)$ "helicity" states:

- **hypermultiplet:** $2(\frac{1}{2},0) + 4(0,0)$, i.e. one complex Weyl fermion and two complex scalars, for a total of 4+4 = 8 real components.
- **vector multiplet:** $(\frac{1}{2},\frac{1}{2}) + 2(0,\frac{1}{2})$, i.e. a massless vector and one complex anti-Weyl fermion, for a total of 4+4 = 8 real components.
- **gravity multiplet:** $(1,1)+2(\frac{1}{2},1)+(0,1)$, i.e. a graviton, two gravitini, and one self-dual 2nd rank antisymmetric tensor, for a total of 9+12+3 = 24 real components.
- **tensor multiplet:** $(1,0)+2(\frac{1}{2},0)+(0,0)$, i.e. one anti-self-dual 2nd rank antisymmetric tensor, one complex Weyl fermion, and one real scalar "dilaton", for a total of 3+4+1 = 8 real components.

Dimensional reduction 6 → 4

Consider six-dimensional N=$(0,1)_8$ supersymmetric Yang-Mills theory. We write the action in terms of fields describing the on-shell massless vector multiplet described above. The action is:

$$\int d^6x \left[-\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}i\bar{\lambda}\gamma^M \nabla_M \lambda\right] \quad , \tag{212}$$

where ∇_M is a gauge covariant derivative.

Now we imagine compactifying this theory on a torus. Let x^4, x^5 be the compactified coordinates. Then the 6-dimensional gauge field A_M breaks up into a 4-dimensional gauge field A_m and two real scalars A_4, A_5. The complex anti-Weyl fermion λ breaks up into two 4-dimensional complex Weyl fermions. In addition we will have an infinite tower of massive Kaluza-Klein states, corresponding to the Fourier decomposition

$$\phi(x^m, x^4, x^5) = \sum_{n_4, n_5} e^{[-in_4 m_4 x^4 - in_5 m_5 x^5]} \phi_{n_4 n_5}(x^m) \quad , \tag{213}$$

where $\phi(x^m, x^4, x^5)$ denotes any 6-dimensional field component, and m_4, m_5 are inversely related to the compactification radii.

The SUSY generator Q_a splits up into two Q_α's, implying that the 4-dimensional theory has N=2 supersymmetry. The 6-dimensional translation generator P_M splits into P_m, P_4, P_5. From the 6-dimensional SUSY algebra we see that P_4 and P_5 commute with the Q_α's and with P_m. They also appear on the right-hand side of $\{Q, Q\}$. Clearly what we have here are two real = one complex central charge:

$$X^{ab} \sim (P_4 + iP_5)\epsilon^{ab} \quad .$$

Multiplying the two Q_αs by a phase to convert to Zumino's basis (see Eq. 35), this corresponds to a single real central charge

$$Z = \sqrt{P_4^2 + P_5^2} \quad . \tag{214}$$

Thus the dimensionally reduced theory consists of 4-dimensional N=2 supersymmetric Yang-Mills with central charges and an infinite number of massive multiplets.

Now suppose we repeat the above excercise, but first adding some massless $d = 6$ hypermultiplets (A, B, ψ) to our N=$(0,1)_8$ supersymmetric Yang-Mills theory. Each $d = 6$ hypermultiplet will give one $d = 4$ multiplet of fields with the counting of the $d = 4$ N=2 massless hypermultiplet, plus a tower of additional massive multiplets. The 6-dimensional on-shell condition for the complex scalars A and B

$$\partial^M \partial_M A = \partial^M \partial_M B = 0 \tag{215}$$

fixes the masses of the 4-dimensional hypermultiplet scalars:

$$4m^2 = Z^2 \quad . \tag{216}$$

Thus the dimensionally reduced theory contains massive BPS-saturated N=2 short multiplets, which as already noted do indeed have the same counting as the $d = 4$ N=2 massless hypermultiplet.

Anomaly cancellation

A striking feature of the 6-dimensional gravity multiplet is that it contains the self-dual part of a 2nd rank antisymmetric tensor, without the anti-self-dual part. As is also the case in four dimensions, it is impossible to write a Lorentz covariant Lagrangian formulation of just the self-dual antisymmetric tensor field. However one can write Lorentz covariant equations of motion, and it appears that the corresponding field theory exists and is Lorentz invariant, despite the lack of a **manifestly** Lorentz invariant action principle. If

$$n_g = n_t = 1 \quad , \tag{217}$$

where n_g, n_t are the number of gravity and tensor multiplets, then of course we can write a Lorentz covariant Lagrangian. Thus we have the interesting result that every 6-dimensional supergravity theory **either** contains a dilaton field **or** has no manifestly Lorentz invariant action principle.

In six dimensions we have both gravitational anomalies and mixed gauge-gravitational anomalies. Anomaly cancellation is a severe constraint on the particle content, and in particular on which combinations of SUSY multiplets yield anomaly-free theories.

For example, when $n_g{=}n_t{=}1$, the necessary and sufficient condition for anomaly cancellation is:

$$n_h = n_v + 244 \quad , \tag{218}$$

where n_h, n_v are the number of hypermultiplets and vector multiplets. Thus we always need a remarkably large number of hypermultiplets to cancel anomalies.

Let's look at two examples of anomaly-free 6-dimensional $N{=}(0,1)_8$ supersymmetric supergravity-Yang-Mills-matter theories.

- Gauge group $E_8 \times E_7$, with 10 massless hypermultiplets in the $(1, 56)$ of $E_8 \times E_7$, and 65 singlet hypermultiplets. Thus:

$$\begin{aligned} n_v &= 248 + 133 = 381 \quad ; \\ n_h &= 560 + 65 = 625 \quad ; \end{aligned} \tag{219}$$

  which satisfies Eq. 218.

- Gauge group $SO(28) \times SU(2)$, with 10 massless hypermultiplets in the $(28, 2)$ of $SO(28) \times SU(2)$, and 65 singlet hypermultiplets. Thus:

$$\begin{aligned} n_v &= 378 + 3 = 381 \quad ; \\ n_h &= 560 + 65 = 625 \quad ; \end{aligned} \tag{220}$$

  which satisfies Eq. 218.

These examples arise as compactifications of the ten-dimensional $E_8 \times E_8$ and $SO(32)$ heterotic strings, respectively, onto the complex dimension 2 Kähler manifold K3.

9.5 *Supersymmetry in 11 dimensions*

Eleven is the maximum dimension in which we can have as few as 32 supersymmetries. Thus $d = 11$ is the maximum dimension of interest to supersymmetry theorists, unless one is willing to make some drastic assumptions such as altering the Minkowski signature of spacetime.

Futhermore, there is only one sensible supersymmetric theory in eleven dimensions: N=1 supergravity. The N=$(1)_{32}$ SUSY algebra is generated by a single Majorana spinor Q_a, $a = 1, \ldots 32$.

Here is the field content of the on-shell massless $d = 11$ N=1 gravity multiplet, characterized by irreps of the little group $SO(9)$:

- e_M^A, the 11-dimensional vielbein. On-shell this constitutes a 44 of $SO(9)$.
- ψ_M^a, $a = 1, \ldots 32$, an 11-dimensional massless vector-spinor, i.e. the gravitino field. On-shell this is a 128 of $SO(9)$.
- A_{MNP}, a 3rd rank antisymmetric tensor. On-shell this is an 84 of $SO(9)$.

The Lagrangian of 11-dimensional supergravity, in terms of these component fields of the on-shell multiplet, is rather simple. [24] The first three terms are:

$$-\frac{1}{2\kappa^2}eR - \frac{1}{2}e\bar{\psi}_M\Gamma^{MNP}D_N\psi_P - \frac{1}{48}eF_{MNPQ}F^{MNPQ} \quad , \tag{221}$$

where:

$$\begin{aligned}
\kappa &= \text{11-dimensional gravitational coupling} \quad , \\
e &\equiv \det e_M^A \quad , \\
R &= \text{Ricci scalar} \quad , \\
\Gamma^{MNP} &= e_A^M e_B^N e_C^P \gamma^{[A}\gamma^B\gamma^{C]} \quad , \\
D_N &= \text{Lorentz covariant derivative for 11-dim. vector-spinors} \quad , \\
F_{MNPQ} &= \partial_{[M}A_{NPQ]}, \text{ i.e. a field strength} \quad ,
\end{aligned}$$

and the square brackets denote antisymmetrization.

Eleven-dimensional supergravity is the field theory limit of M-theory, which is in turn a strong coupling limit of superstring theory.

Dimensional reductions of $d = 11$ supergravity

We can truncate $d = 1$ supergravity down to a 4-dimensional theory by simply suppressing the dependence on x^4-x^{10}. Since the resulting 4-dimensional theory still has 32 supersymmetries, we obviously must get $N=(8)_{32}$ extended supergravity. Provided one is satisfied with on-shell formulations, this is a simpler way of deriving the rather complicated 4-dimensional theory.

Another useful example is to truncate $d = 11$ to $d = 10$. The Majorana spinor Q_a (32 components) splits into two 10-dimensional Majorana-Weyl spinors Q_α, $\bar{Q}_{\dot\alpha}$ (16 components each) with opposite chirality. Thus the truncated theory is a nonchiral $d = 10$ $N=(1,1)_{32}$ supergravity, commonly known as **Type IIA supergravity.** [25]

The components of the 11-dimensional vielbein break up as follows:

$$e_M^A \rightarrow \begin{pmatrix} e_M^A & A_M \\ 0 & \phi \end{pmatrix} \quad , \tag{222}$$

where we have set the 1×10 block on the lower left to zero using the freedom of those local Lorentz transformations which mix the x^{11} direction with the other ten. The 10-dimensional massless vector A_M arises from the e_M^{11} components of the vielbein, while the real scalar ϕ (the 10-dimensional dilaton) arises from the e_{11}^{11} component.

The 11-dimensional antisymmetric tensor field A_{MNP} splits into a 10-dimensional 3rd rank antisymmetric tensor A_{MNP} and a 2nd rank antisymmetric tensor B_{MN} (from A_{MN11}). The 11-dimensional gravitino field ψ_M^a becomes:

$$\psi_M^a \rightarrow \psi_M^\alpha,\ \bar{\psi}_M^{\dot\alpha},\ \lambda^\alpha,\ \bar{\lambda}^{\dot\alpha} \quad ; \tag{223}$$

giving two 10-dimensional Majorana-Weyl vector-spinors of opposite chirality, and two 10-dimensional Majorana-Weyl spinors of opposite chirality.

Since Type IIA supergravity is vectorlike it is trivially free of gravitational anomalies. It is the field theory limit of the Type IIA superstring.

9.6 More on supersymmetry in 10 dimensions

Type IIA supergravity has $N=(1,1)_{32}$ supersymmetry. Since 16 is the minimum spinor dimension in ten dimensions, we can in principle construct a chiral $N=(1,0)_{16}$ supergravity theory as well; however such a theory has gravitational anomalies. The only other possibility in ten dimensions is a chiral $N=(2,0)_{32}$ supergravity. This theory, known as **Type IIB supergravity**, turns out to be anomaly-free, due to highly nontrivial cancellations. Type IIB cannot be obtained by dimensional reduction of 11-dimensional supergravity.

Table 3: Comparison of Type II supergravities.

Type IIA:	1	28	35_v	8_v	56_v	
	ϕ	B_{MN}	e_M^A	A_M	A_{MNP}	
	8_c	8_s	56_s	56_c		
	λ	$\tilde{\lambda}$	ψ	$\tilde{\psi}$		
Type IIB:	1	28	35_v	1	28	35_c
	A	A_{MN}	e_M^A	$\tilde{A}$	$\tilde{A}_{MN}$	A_{MNPQ}
	8_s	8_s	56_s	56_s		
	λ	$\tilde{\lambda}$	ψ	$\tilde{\psi}$		

N=$(2,0)_{32}$ supersymmetry is generated by two Majorana-Weyl spinors Q_α, $\tilde{Q}_\alpha$ of the same chirality. The on-shell Type IIB supergravity multiplet has the following field content:

- e_M^A, the 10-dimensional vielbein.
- A_{MNPQ}, a **self-dual** 4th rank antisymmetric tensor.
- A_{MN}, $\tilde{A}_{MN}$, 2nd rank antisymmetric tensors.
- A, $\tilde{A}$, real scalars.
- ψ_M^α, $\tilde{\psi}_M^\alpha$, Majorana-Weyl vector-spinors of the same chirality.
- $\lambda^{\dot{\alpha}}$, $\tilde{\lambda}^{\dot{\alpha}}$, Majorana-Weyl spinors of the same chirality.

Because the field content includes the self-dual part of a 4th rank antisymmetric tensor, Type IIB supergravity does not have a Lorentz covariant Lagrangian formulation.

The little group in 10-dimensions is $SO(8)$. Because of the special automorphism symmetry of the Lie algebra D_4, all of the dimension<224 irreps except the adjoint occur as triplets of irreps with the same dimension and index. These irreps are distinguished by subscripts v, s, and c. Thus we can gain more information about the differences between Type IIA and Type IIB supergravity by listing the $SO(8)$ irreps corresponding to each component field. Note that both theories have $128+128=256$ real field components.

Let us consider again 10-dimensional $N=(1,0)_{16}$ supergravity, also known as **Type I supergravity**. As we have noted, this theory is anomalous. Remarkably though, by coupling this theory to the 10-dimensional supersymmetric Yang-Mills multiplet, we can in certain cases obtain theories free of both gravitational and mixed gauge-gravitational anomalies. The field content of the chiral supergravity is just a truncation of the Type IIA fields:

$$\begin{array}{l} e_M^A,\ B_{MN},\ \phi \quad ; \\ \psi_M^\alpha,\ \bar{\lambda}^{\dot{\alpha}} \quad ; \end{array}$$

for a total of 64+64 = 128 real field components. The 10-dimensional on-shell $N=(1,0)_{16}$ supersymmetric Yang-Mills multiplet consists of a massless vector and a massless Majorana-Weyl spinor

$$A_M,\ \chi^\alpha \quad ;$$

in the 8_v and 8_c irreps of the little group, and the adjoint representation of some gauge group.

The details [26] of how to couple N=(1,0) supergravity to N=(1,0) Yang-Mills, consistent with both local supersymmetry and gauge invariance, is beyond the scope of these lectures. We merely note that one surprising result is that the gauge invariant field strength of the massless antisymmetric tensor field B_{MN} has an extra contribution which is a functional of the Yang-Mills gauge field A_M. Then, even though B_{MN} is a gauge singlet, gauge invariance requires that B_{MN} transforms nontrivially under Yang-Mills gauge transformations. This strange fact makes possible the Green-Schwarz anomaly cancellation mechanism.

The coupled $d = 10$ supergravity-Yang-Mills theory is anomaly-free only for the following choices of gauge group:

$$SO(32),\ E_8 \times E_8,\ E_8 \times [U(1)]^{248},\ [U(1)]^{496} \quad .$$

The first two choices correspond to the field theory limits of the $SO(32)$ and $E_8 \times E_8$ heterotic strings; the other two choices can probably also be connected to superstring theory using D-brane arguments.

10 Conclusion

It is a remarkable fact that many technical SUSY topics, thought until recently to be of purely academic interest, have turned out to be crucial to obtaining deep insights about the physics of strongly-coupled nonabelian gauge theories and strongly-coupled string theory. Even if there is no weak scale SUSY in the real world – even if there is no SUSY at all – supersymmetry has earned its place in the pantheon of Really Good Ideas. Furthermore, we are encouraged to continue to develop and expand the technical frontiers of supersymmetry, confident both that there is still much to learn, and that this new knowledge will find application to important physical problems.

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Appendix

Notation and conventions

My notation and conventions in these lectures conforms with Wess and Bagger[2] **with the following exceptions:**

- I use the standard "West Coast" metric:

$$\eta^{mn} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{224}$$

  This is the standard metric convention of particle physics. Wess and Bagger use the East Coast metric still popular with relativists and other benighted souls.

  Changes which follow from my different choice of metric:

  1. My σ^0 and $\bar{\sigma}^0$ are the 2x2 identity matrix instead of minus it.
  2. I define the SL(2,C) generators with an extra factor of i:

$$\sigma_\alpha^{mn\beta} = \frac{i}{4}\left[\sigma^m_{\alpha\dot{\gamma}}\bar{\sigma}^{n\dot{\gamma}\beta} - \sigma^n_{\alpha\dot{\gamma}}\bar{\sigma}^{m\dot{\gamma}\beta}\right] \quad . \tag{225}$$

  3. I define the Levi-Civita tensor density $\epsilon_{0123} = +1$ instead of minus one.

4. As a result of this, σ_{mn} is self-dual and $\bar{\sigma}_{mn}$ is anti-self-dual, instead of vice-versa.

5. In the full component expansion of the chiral superfield $\Phi(x, \theta, \bar{\theta})$, the $(\theta\theta)(\bar{\theta}\bar{\theta})$ term comes in with the opposite sign.

Spinor definitions and identities

Irreps of SL(2,C) $\approx$ SO(1,3):

$$
\begin{aligned}
(\tfrac{1}{2}, 0) &= \text{left-handed 2 component Weyl spinor} \\
(0, \tfrac{1}{2}) &= \text{right-handed 2 component Weyl spinor}
\end{aligned}
$$

In Van der Waerden notation, undotted $= (\frac{1}{2}, 0)$, dotted $= (0, \frac{1}{2})$:

$$
\begin{aligned}
(\tfrac{1}{2}, 0) &\quad : \quad \psi_\alpha \quad , \\
(0, \tfrac{1}{2}) &\quad : \quad \bar{\psi}^{\dot{\alpha}} \equiv (\psi_\alpha)^* \quad .
\end{aligned}
$$

Also:

$$
\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^\dagger \quad ; \qquad \psi^\alpha \equiv (\bar{\psi}_{\dot{\alpha}})^* \quad . \tag{226}
$$

We raise and lower spinor indices with the 2-dimensional Levi-Civita symbols:

$$
\begin{aligned}
\epsilon_{\alpha\beta} = \quad \epsilon_{\dot{\alpha}\dot{\beta}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad ; \\
\epsilon^{\alpha\beta} = \quad \epsilon^{\dot{\alpha}\dot{\beta}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \quad .
\end{aligned} \tag{227}
$$

Thus:

$$
\begin{aligned}
\psi^\alpha &= \epsilon^{\alpha\beta}\psi_\beta \quad ; \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \quad ; \\
\bar{\psi}^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \quad ; \qquad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}} \quad .
\end{aligned} \tag{228}
$$

Note also:

$$
\begin{aligned}
\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} &= \delta_\alpha^\gamma \quad , \\
\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} &= \delta_{\dot\alpha}^{\dot\gamma} \quad , \\
\epsilon_{\alpha\beta}\epsilon^{\delta\gamma} &= \delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma \quad , \\
\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\delta\dot\gamma} &= \delta_{\dot\alpha}^{\dot\gamma}\delta_{\dot\beta}^{\dot\delta} - \delta_{\dot\alpha}^{\dot\delta}\delta_{\dot\beta}^{\dot\gamma} \quad .
\end{aligned}
\tag{229}
$$

Pauli Matrices:

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{230}
$$

From which we define:

$$
\begin{aligned}
\sigma^m &= (I, \vec\sigma) &= \bar\sigma_m \quad , \\
\bar\sigma^m &= (I, -\vec\sigma) &= \sigma_m \quad ,
\end{aligned}
\tag{231}
$$

where I denotes the 2x2 identity matrix. Note that in these definitions "bar" **does not** indicate complex conjugation.

σ^m has undotted-dotted indices: $\sigma^m_{\alpha\dot\beta}$

$\bar\sigma^m$ has dotted-undotted indices: $\bar\sigma^{m\dot\alpha\beta}$

We also have the completeness relations:

$$
\begin{aligned}
\mathrm{tr}\,\sigma^m\bar\sigma^n &= 2\eta^{mn} \quad , \\
\sigma^m_{\alpha\dot\beta}\bar\sigma_m^{\dot\gamma\delta} &= 2\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma} \quad .
\end{aligned}
\tag{232}
$$

σ^m and $\bar\sigma^m$ are related by the Levi-Civita symbols:

$$
\bar\sigma^{m\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}\sigma^m_{\delta\dot\gamma} \quad ; \quad \sigma^m_{\alpha\dot\beta} = \epsilon_{\dot\delta\dot\beta}\epsilon_{\gamma\alpha}\bar\sigma^{m\dot\delta\gamma} \quad . \tag{233}
$$

It is occasionally convenient to do a "fake" conversion of an undotted to a dotted index or vice versa using the fact that σ^0 and $\bar\sigma^0$ are just the identity matrix:

$$
\psi^\alpha = (\bar\psi_{\dot\beta})^*\bar\sigma^{0\dot\beta\alpha} \quad ; \quad \bar\psi^{\dot\alpha} = (\psi_\beta)^*\sigma^{0\beta\dot\alpha} \quad . \tag{234}
$$

Because the Pauli matrices anticommute. i.e.

$$\{\sigma^i, \sigma^j\} = \delta^{ij} I \qquad i, j = 1, 2, 3 \tag{235}$$

we have the relations:

$$\begin{aligned}
(\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m)_\alpha^{\ \beta} &= 2\eta^{mn} \delta_\alpha^\beta \quad , \\
(\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)^{\dot{\alpha}}_{\ \dot{\beta}} &= 2\eta^{mn} \delta^{\dot{\alpha}}_{\dot{\beta}} \quad .
\end{aligned} \tag{236}$$

Spinor Summation Convention:

$$\begin{aligned}
\psi\chi &= \psi^\alpha \chi_\alpha &&= -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi \\
\bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} &&= -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi}
\end{aligned} \tag{237}$$

Note that these quantities are Lorentz scalars.
We also have:

$$\begin{aligned}
(\chi\psi)^\dagger &= (\chi^\alpha \psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \\
(\chi \sigma^m \bar{\psi})^\dagger &= \psi \sigma^m \bar{\chi} = \text{Lorentz vector}
\end{aligned} \tag{238}$$

Other useful relations are:

$$\begin{aligned}
\psi^\alpha \psi^\beta &= -\frac{1}{2} \epsilon^{\alpha\beta} \psi\psi \quad , \\
\psi_\alpha \psi_\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \psi\psi \quad , \\
\bar{\psi}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}} &= \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}\bar{\psi} \quad , \\
\bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} &= -\frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}\bar{\psi} \quad .
\end{aligned} \tag{239}$$

The SL(2,C) generators are defined as

$$\begin{aligned}
\sigma^{mn\beta}_{\alpha} &= \frac{i}{4} \left[\sigma^m_{\alpha\dot{\gamma}} \bar{\sigma}^{n\dot{\gamma}\beta} - \sigma^n_{\alpha\dot{\gamma}} \bar{\sigma}^{m\dot{\gamma}\beta} \right] \\
\bar{\sigma}^{mn\dot{\alpha}}{}_{\dot{\beta}} &= \frac{i}{4} \left[\bar{\sigma}^{m\dot{\alpha}\gamma} \sigma^n_{\gamma\dot{\beta}} - \bar{\sigma}^{n\dot{\alpha}\gamma} \sigma^m_{\gamma\dot{\beta}} \right]
\end{aligned} \tag{240}$$

We have:

$$\begin{array}{lll} \epsilon^{mnpq}\sigma_{pq} = 2i\sigma^{mn} & : & \text{self dual } (1,0) \\ \epsilon^{mnpq}\bar{\sigma}_{pq} = -2i\bar{\sigma}^{mn} & : & \text{anti self dual } (0,1) \end{array} \tag{241}$$

And thus the trace relation:

$$\mathrm{tr}\,[\sigma^{mn}\sigma^{pq}] = \frac{1}{2}(\eta^{mp}\eta^{nq} - \eta^{mq}\eta^{np}) + \frac{i}{2}\epsilon^{mnpq} \quad . \tag{242}$$

Fierz identities

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\theta\theta)(\phi\psi) \tag{243}$$

$$(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\frac{1}{2}(\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}) \tag{244}$$

$$\phi\sigma^m\bar{\chi} = -\bar{\chi}\bar{\sigma}^m\phi \tag{245}$$

$$\phi\sigma_m\bar{\chi} = -\bar{\chi}\bar{\sigma}_m\phi \tag{246}$$

$$(\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta}) = \frac{1}{2}\eta^{mn}(\theta\theta)(\bar{\theta}\bar{\theta}) \tag{247}$$

$$(\sigma^m\bar{\theta})_\alpha(\theta\sigma^n\bar{\theta}) = \frac{1}{2}\eta^{mn}\theta_\alpha(\bar{\theta}\bar{\theta}) - i(\sigma^{mn}\theta)_\alpha(\bar{\theta}\bar{\theta}) \tag{248}$$

$$(\theta\phi)(\bar{\theta}\bar{\psi}) = \frac{1}{2}(\theta\sigma^m\bar{\theta})(\phi\sigma_m\bar{\psi}) \tag{249}$$

$$(\bar{\theta}\bar{\psi})(\theta\phi) = \frac{1}{2}(\theta\sigma^m\bar{\theta})(\phi\sigma_m\bar{\psi}) = (\theta\phi)(\bar{\theta}\bar{\psi}) \tag{250}$$

General 4-dimensional SUSY algebra

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A_B \tag{251}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} a^{\ell AB} B_\ell \tag{252}$$

$$\{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = -\epsilon_{\dot\alpha\dot\beta} a^*_{\ell AB} B^\ell \tag{253}$$

$$[Q^A_\alpha, P_m] = [\overline{Q}^A_{\dot\alpha}, P_m] = 0 \tag{254}$$

$$[Q^A_\alpha, M_{mn}] = \sigma_{mn\alpha}{}^\beta Q^A_\beta \tag{255}$$

$$[\overline{Q}^{\dot\alpha}_A, M_{mn}] = \bar\sigma^{\dot\alpha}_{mn\dot\beta} \overline{Q}^{\dot\beta}_A \tag{256}$$

$$[P_m, P_n] = 0 \tag{257}$$

$$[M_{mn}, P_p] = i(\eta_{np} P_m - \eta_{mp} P_n) \tag{258}$$

$$\begin{aligned}[M_{mn}, M_{pq}] = &-i\big(\eta_{mp} M_{nq} - \eta_{mq} M_{np} \\ &-\eta_{np} M_{mq} + \eta_{nq} M_{mp}\big)\end{aligned} \tag{259}$$

$$[Q^A_\alpha, B_\ell] = S^A_{\ell\ B} Q^B_\alpha \tag{260}$$

$$[\overline{Q}_{\dot\alpha A}, B_\ell] = -S^*_{\ell A}{}^B \overline{Q}_{\dot\alpha B} \tag{261}$$

$$[B_\ell, B_k] = iC_{\ell k}{}^j B_j \tag{262}$$

$$[P_m, B_\ell] = [M_{mn}, B_\ell] = 0 \tag{263}$$

Where the a^ℓ are antisymmetric matrices, and S_ℓ, a_ℓ must satisfy the intertwining relation:

$$S^A_{\ell\ C}\, a^{CBk} = -a^{ACk}\, S^{*\ell B}_C \tag{264}$$

Note also the perverse but essential convention implicit in Wess and Bagger:

$$a^{\ell AB} \quad = \quad -a^{\ell}_{AB} \tag{265}$$

N=1 SUSY algebra in 4 dimensions

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} \quad = \quad 2\sigma^m_{\alpha\dot\beta} P_m \tag{266}$$

$$\{Q_\alpha, Q_\beta\} \quad = \quad \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} \quad = \quad 0 \tag{267}$$

$$[Q_\alpha, P_m] \quad = \quad [\overline{Q}_{\dot\alpha}, P_m] \quad = \quad 0 \tag{268}$$

$$[Q_\alpha, M_{mn}] \quad = \quad \sigma_{mn\alpha}{}^{\beta} Q_\beta \tag{269}$$

$$[\overline{Q}^{\dot\alpha}, M_{mn}] \quad = \quad \bar\sigma^{\dot\alpha}_{mn\dot\beta} \overline{Q}^{\dot\beta} \tag{270}$$

$$[P_m, P_n] \quad = \quad 0 \tag{271}$$

$$[M_{mn}, P_p] \quad = \quad i(\eta_{np}P_m - \eta_{mp}P_n) \tag{272}$$

$$[M_{mn}, M_{pq}] \quad = -i\,(\eta_{mp}M_{nq} - \eta_{mq}M_{np} - \eta_{np}M_{mq} + \eta_{nq}M_{mp}) \tag{273}$$

$$[Q_\alpha, R] \quad = \quad RQ_\alpha \tag{274}$$

$$[\overline{Q}_{\dot\alpha}, R] \quad = \quad -R\overline{Q}_{\dot\alpha} \tag{275}$$

$$[P_m, R] \quad = \quad [M_{mn}, R] \quad = \quad 0 \tag{276}$$

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